CSE520: Computational Geometry
Lecture 5
Topological Lower Bounds II

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June 15, 2020

1. Introduction
2. Set disjointness
3. Optimization problems
4. Diameter
5. Maximum gap
6. Discussion

Outline

- In this lecture, we give more example of lower bounds based on Ben-Or's theorem:
  - Set disjointness.
  - Diameter of a point-set.
  - Maximum gap.

References:

- Textbook by Preparata and Shamos.
- Dave Mount's lecture notes, Lecture 26.
- Ben-Or's paper.

Set Disjointness

Problem (Set Disjointness)

Given two real n-tuples $(a_1, \ldots, a_n) \in \mathbb{R}^n$ *and* $(b_1, \ldots, b_n) \in \mathbb{R}^n$*, determine whether there exists a pair* i, j *such that* $a_i = b_j$*.*

Theorem

Any ACT that solves the set disjointness problem has height $\Omega(n \log n)$*.*

- We now prove this theorem. To this end, we introduce the problem below, which is the special case where $(b_1, \ldots, b_n) = (1, \ldots, n)$.
- We denote $\mathbb{N}_n = \{1, \ldots, n\}$.

Problem (1)

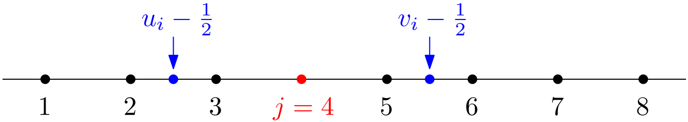
Given $(a_1, \ldots, a_n) \in \mathbb{R}^n$*, decide whether there exists* i *such that* $a_i \in \mathbb{N}_n$*.*

Set Disjointness

- We will give an $\Omega(n \log n)$ lower bound for Problem 1. As it is a special case of set disjointness, it implies the same lower bound for set disjointness. (See argument in previous lecture on line segment intersection detection.)

- Let W^- be the set of *negative* instances of Problem 1.
- So it is the set of tuples $(a_1, \ldots, a_n)$ such that $a_i \notin \mathbb{N}_n$ for all n.
- Let $u : \mathbb{N}_n \to \mathbb{N}_n$ be an arbitrary function from $\mathbb{N}_n$ into $\mathbb{N}_n$.
- Let $a(u) = (u_1 - \frac{1}{2}, u_2 - \frac{1}{2}, \ldots, u_n - \frac{1}{2})$.
- Let $v \neq u$ be another function $v : \mathbb{N}_n \to \mathbb{N}_n$.
- Then $a(u)$ and $a(v)$ are in W^-.
- We now show that they are in different connected components of W^-.

Set Disjointness



- As $u \neq v$, there exists i such that $u_i \neq v_i$.
- So there must be an integer $j \in \{1, \ldots, n-1\}$ between $u_i - \frac{1}{2}$ and $v_i - \frac{1}{2}$.
- So for any path γ from $a(u)$ to $a(v)$, there exists $t \in [0,1]$ such that $\gamma_i(t) = j$.
- Therefore, $\gamma(t) \notin W^-$, so γ is not contained in W^-.
- It follows that $a(u)$ and $a(v)$ lie in different connected components of W^-.

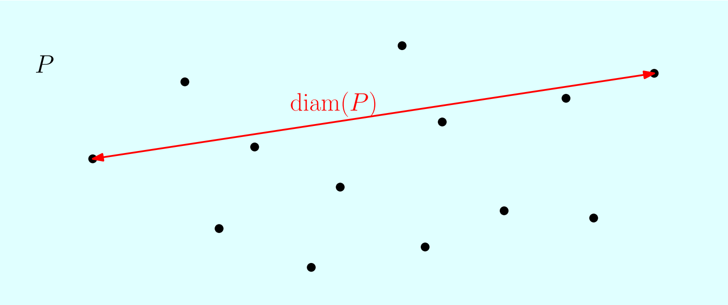
Set Disjointness

- The number of functions $u : \mathbb{N}_n \to \mathbb{N}_n$ is n^n.
- So there are n^n points $a(u)$, and thus n^n connected components in W^-.
- As $\log(n^n) = n \log n$, By Ben-Or's theorem, it follows that:

Lemma

Any algebraic computation tree deciding Problem 1 has height $\Omega(n \log n)$.

Diameter



Problem (Diameter of a Point-Set)

The diameter $\mathrm{diam}(P)$ *of a set* $P = \{p_1, \ldots, p_n\}$ *of* n *points is the maximum distance between any two points:*

$$\mathrm{diam}(P) = \max_{i,j} \mathrm{d}(p_i, p_j).$$

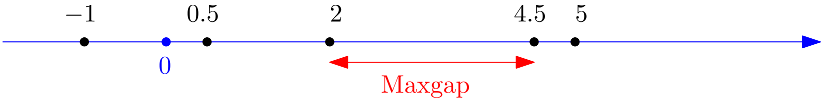
Maximum Gap

Problem (maximum gap)

Given a set of n (unsorted) numbers, the maximum gap problem is to find the largest gap between two consecutive numbers in sorted order.

Example

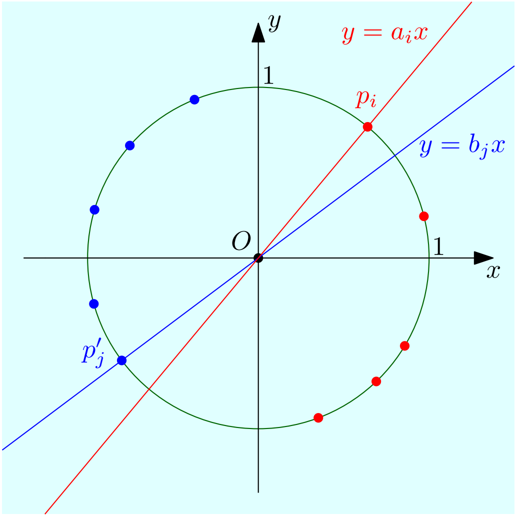
- INPUT: 0.5, 2, 5, 4.5, -1
- OUTPUT: 2.5



Optimization Algorithms

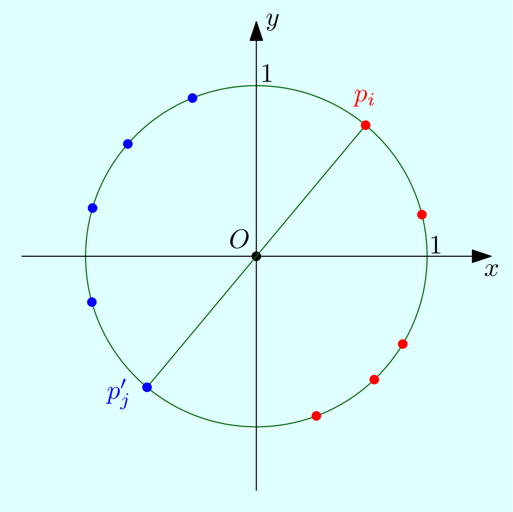
- The ACT model deals with *decision* problems: problems with a YES-NO answer.
- We will abuse notation and use ACTs for *optimization* problems:
  - The 2D-diameter problem.
  - Maximum gap.
- In fact, our lower bound argument will be for the associated decision problems: For some $\delta \in \mathbb{R}$,
  - Is the diameter at least δ?
  - Is the maximum gap at most δ?
- An optimization problem is at least as hard as the corresponding decision problem, because once you have the optimal value, you can compare it with δ in $O(1)$ time.
- So a lower bound on the decision problem implies a lower bound on the optimization problem.

Diameter



Given an instance A, B of set disjointness, we map A to the right side and B to the left side of the unit circle, respectively.

Diameter



The diameter is 2 iff two points p_i, p'_j are diametrically opposed, that is $a_i = b_j$.

Diameter

Theorem

In the ACT model, the complexity of the 2D-diameter problem is $\Omega(n \log n)$*.*

Proof.

An instance of set disjointness is mapped to the unit circle as follows:

- Each a_i is mapped to the point $p_i = (x_i, y_i)$ such that $y_i = a_i x_i$, $x_i > 0$, and $x_i^2 + y_i^2 = 1$.
- Each b_j is mapped to the point $p'_j = (x'_j, y'_j)$ such that $y_j = b_j x_j$, $x_j < 0$, and $x_j^2 + y_j^2 = 1$.

Then the diameter of $\{p_1, \dots, p_n, p'_1, \dots, p'_n\}$ is 2 iff $p_i = -p'_j$ for some i, j, which means that $a_i = b_j$ and hence $A \cap B \neq \emptyset$. $\square$

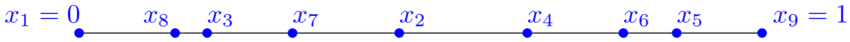
Remark: it is crucial in this argument that the coordinates of p_i and p'_i can be computed in constant time by an ACT.

Maximum Gap

Theorem

In the ACT model, the maximum gap problem has complexity $\Omega(n \log n)$.

- We now prove this theorem.
- Again, we prove it for a special case:



Problem (2)

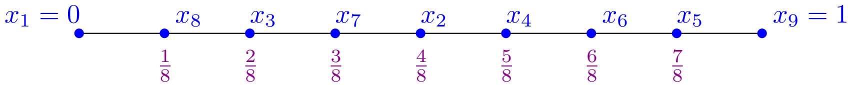
Given a n-tuple $(x_1, \ldots, x_n)$ numbers in $[0, 1]$, such that $x_1 = 0$ and $x_n = 1$, decide whether the maximum gap of X is $\frac{1}{n-1}$.

Maximum Gap

Lemma

Any ACT deciding Problem 2 has height $\Omega(n \log n)$.

- We now prove this lemma.
- What do the positive instances look like?



- In order for the maximum gap to be $1/(n-1)$, all the points must be regularly spaced, with spacing $1/(n-1)$.
- So $(x_2, \ldots, x_{n-1})$ is a permutation of $\left(\frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}\right)$.

Maximum Gap

- So W^- is the set of point of $(0, x_2, x_3, \dots, x_{n-1}, 1)$ where $(x_2, \dots, x_{n-1})$ is a permutation of $\left(\frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}\right)$.
- How many connected components does it have?
- $(n-2)!$ because it is a set of $(n-2)!$ points.
- As $\log((n-2)!) = \Theta((n-2)\log(n-2)) = \Theta(n \log n)$, Problem 1 requires $\Omega(n \log n)$ time in the ACT model.

Maximum Gap

- How fast can we solve maximum gap?
- $O(n \log n)$-time algorithm:

Pseudocode

```
procedure MAXGAP(x_1,...,x_n)
    (y_1,...,y_n) ← Mergesort(x_1,...,x_n)        ▷ O(n log n) time
    M = y_2 − y_1
    for i ← 3, n do                               ▷ O(n) time.
        if y_i − y_{i−1} > M then
            M ← y_i − y_{i−1}
    return M
```

- Conclusion: the complexity of maximum gap in the ACT model is $\Theta(n \log n)$.

Maximum Gap

- However, there is a *linear-time* algorithm.
- Approach: bucketing.

- First we reduce to the case where $\min_i x_i = 1$ and $\max_i x_i = n$.
- So we first compute $m = \min_i x_i$ and $M = \max_i x_i$ in time $O(n)$:

Pseudocode

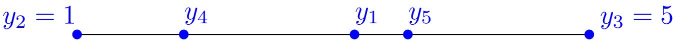
```
procedure MAX(x_1, ..., x_n)
    M ← x_1
    for i ← 2, n do
        if x_i > M then
            M ← x_i
    return M
```

Maximum Gap

- So we replace each x_i with

$$y_i = 1 + (n-1)\frac{x_i - m}{M - m},$$

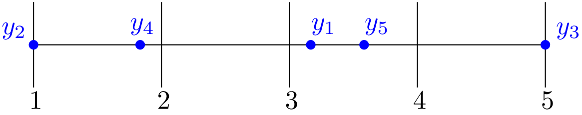
- we solve the maximum gap problem for the $y_1, \dots, y_n$,
- and we multiply the result by $\frac{M-m}{n-1}$



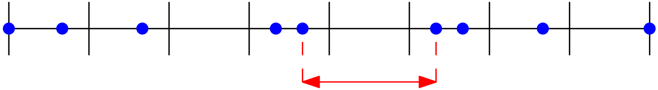
- We now show how compute the maximum gap for the y_i's.

Maximum Gap

- We first make n buckets: $[1,2)$, $[2,3)$, $\ldots$, $[n-1,n)$, $\{n\}$.



- Observation: the maximum gap is at least 1 because the sum of the gap lengths is $n-1$, and there are $n-1$ gaps.
- So the maximum gap is between two points in different buckets.
- Therefore the maximum gap is between the largest number in a bucket and the smallest in the next non-empty bucket.



Maximum Gap

Computing the maximum gap in $O(n)$ time

```
procedure MAXGAPBYBUCKETING(y_1,...,y_n)
    for i ← 1, n do
        store y_i in bucket ⌊y_i⌋
    compute the minimum m_j of each bucket j
    compute the maximum M_j of each bucket j
    result ← 0, last ← M_1
    for j ← 2, n do
        if bucket j is nonempty then
            if m_j − last > result then
                result ← m_j − last
            last ← M_j
    return result
```

Discussion

It would seem that the ACT model is not powerful enough, as the lower bound for maxgap can be broken using the floor function.

- However, it is not reasonable to allow the use of the floor function: It has been proved that the floor function together with $+, -, \times, /$ allows to solve NP-hard problems in polynomial time. (Schönhage, On the power of random access machines, ICALP 79.)

In fact, the ACT model is very powerful:

- It allows *exact* computation in constant time per arithmetic operation.
- Even with high degree polynomials, or square roots.