CSE520: Computational Geometry
Lecture 9
Segment Trees

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Outline

- A new data structure: the segment tree.
- Applications:
  - Stabbing queries.
  - Rectangle intersection.
- Generalization to higher dimension.

Reference:

- Textbook Chapter 10.
- Dave Mount's lecture notes, Lecture 13.

Stabbing Queries

- Orthogonal range searching: data points, query rectangle.
- Stabbing problem: data rectangles, query point.

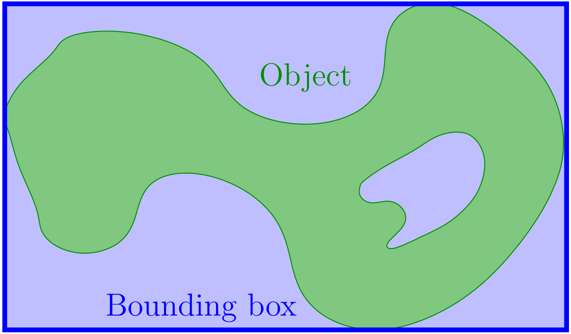
Problem (One-dimensional stabbing problem)

Preprocess a set of n intervals so as to be able to report quickly the k intervals that contain a query number q.

- In $\mathbb{R}^d$:
  - INPUT: a set of n boxes, a query point q.
  - OUTPUT: the k boxes that contain q.

Motivation

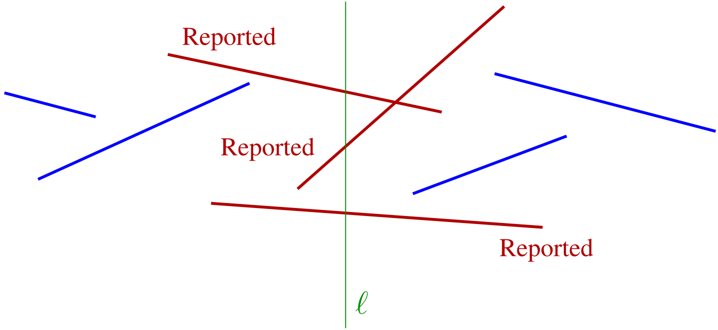
- In graphics and databases, geometric objects are often approximated by their bounding box.



- Query: Which objects does point x belong to?
- First find objects whose bounding boxes intersect x.

Segment Trees

- A data structure to store intervals of $\mathbb{R}$, or segments in $\mathbb{R}^2$.
- Allows us to answer stabbing queries.
  - In $\mathbb{R}^2$: Report the segments that intersect a query vertical line ℓ.



  - Query time: $O(k + \log n)$.
  - Space usage: $O(n \log n)$.
  - Preprocessing time: $O(n \log n)$.

Notation

- Let $S = (s_1, s_2, \ldots, s_n)$ be a set of segments in $\mathbb{R}^2$.
- Let E be the set of the x-coordinates of the endpoints of the segments of S.
- First sort E:

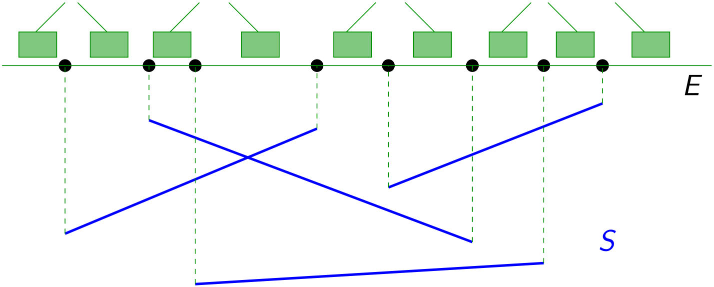
$$E = \{e_1, e_2, \ldots, e_m\},$$

$$e_1 < e_2 < \cdots < e_m.$$

  - $m \leqslant 2n$, with equality in general position.

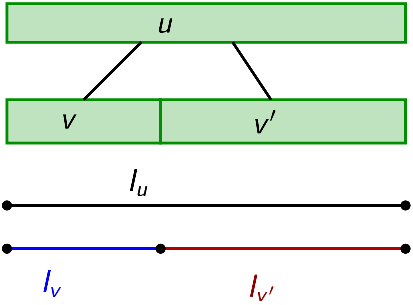
Atomic Intervals

- E splits $\mathbb{R}$ into $m+1$ *atomic intervals:*
  - $(-\infty, e_1]$
  - $[e_i, e_{i+1}]$ for $i \in \{1, 2, \dots m-1\}$
  - $[e_m, \infty)$
- These intervals are stored at the leaves of the segment tree.



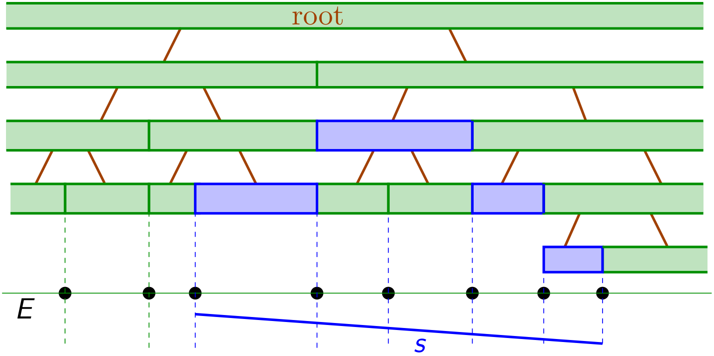
Internal Nodes

- The segment tree $\mathcal{T}$ is a balanced binary search tree.
- Each internal node u with children v and v' is associated with an interval $I_u = I_v \cup I'_v$.
- An *elementary interval* is an interval associated with a node of $\mathcal{T}$. (It can be an atomic interval).



Partitioning a Segment

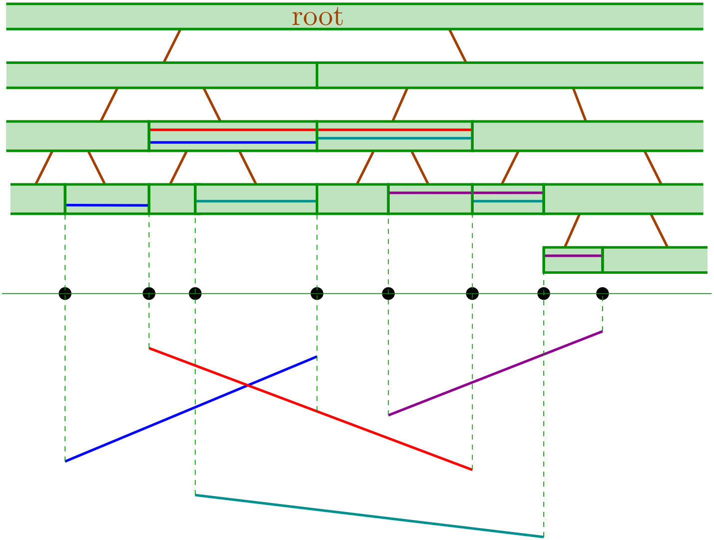
- Let $s \in S$ be a segment whose endpoints have x-coordinates e_i, e_j.
- $[e_i, e_j]$ is split into several elementary intervals.
- These intervals are chosen as close as possible to the root.
- Segment s is stored at each node associated with these elementary intervals.



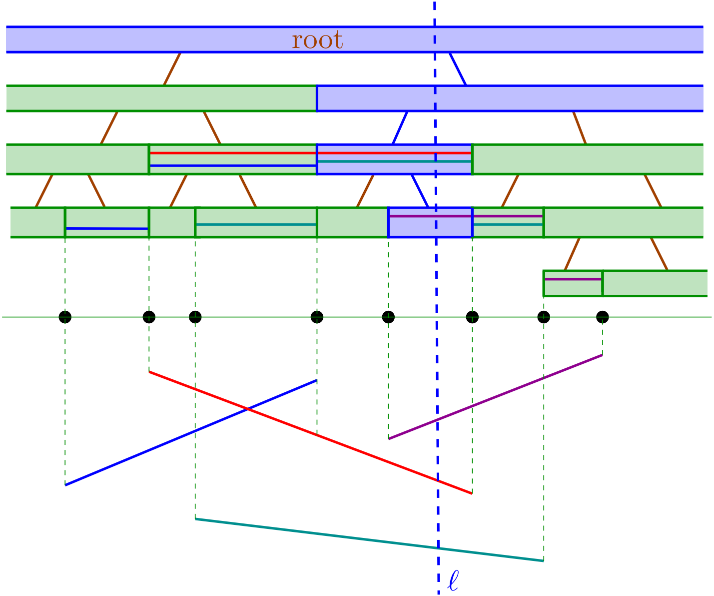
Standard Lists

- At each node u, we store a *standard list* of segments L_u.
- Let $e_i < e_j$ be the x-coordinates of the endpoints of $s \in S$.
- Then s is stored in L_u iff $I_u \subset [e_i, e_j]$ and $I_{\text{parent}(u)} \not\subset [e_i, e_j]$.
  (See previous slide and next slide.)

Example



Answering a Stabbing Query



Pseudocode

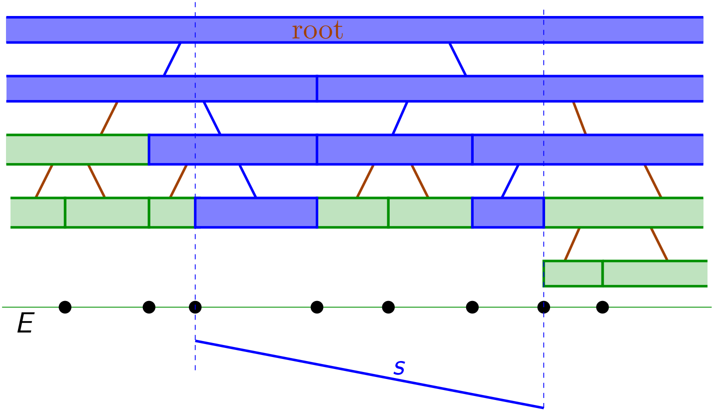
Answering a stabbing query

Algorithm $\mathit{ReportStabbing}(u, x_\ell)$
Input: The root u of $\mathcal{T}$, the x-coordinate x_ℓ of ℓ.
Output: The segments in S that cross ℓ.

Report L_u
if u is an internal node
    then if $x_\ell \in I_{u.left}$
        then $\mathit{ReportStabbing}(u.left, x_\ell)$.
        if $x_\ell \in I_{u.right}$
        then $\mathit{ReportStabbing}(u.right, x_\ell)$.

- Query time: $O(k + \log n)$.

Inserting a Segment



Pseudocode

Inserting a segment into a segment tree

Algorithm *Insert*(u, s)
Input: The root u of $\mathcal{T}$, a segment $s = ((x_1, y_1), (x_2, y_2))$.

if $I_u \subset [x_1, x_2]$
  then $L_u \leftarrow L_u \cup \{s\}$
  else
    if $(x_1, x_2] \cap I_{u.left} \neq \emptyset$
      then $Insert(u.left, s)$
    if $[x_1, x_2) \cap I_{u.right} \neq \emptyset$
      then $Insert(u.right, s)$

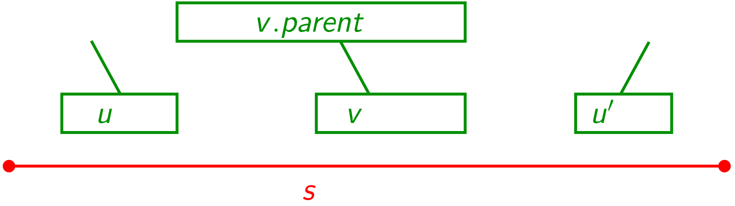
Property

Property

Any segment s is stored at most twice at each level of the segment tree $\mathcal{T}$.

Proof (by contradiction):

- Suppose that s is stored at more than 2 nodes at level i.
- Let u be the leftmost such node, u' be the rightmost.
- Let v be another node at level i containing s.



- Then $I_{v.parent} \subset [x_1, x_2]$.
- So s cannot be stored at v.

Analysis

The property in previous slide implies:

- Space usage $O(n \log n)$.
    - Actually space usage is $\Theta(n \log n)$. (Example?)
- Insertion in $O(\log n)$ time.
  (Similar proof: four nodes at most are visited at each level).
- Query time $O(k + \log n)$.
- Preprocessing:
    - Sort endpoints in $\Theta(n \log n)$ time.
    - Build empty segment tree over these endpoints in $O(n)$ time.
    - Insert n segments into $\mathcal{T}$ in $O(n \log n)$ time.
    - Overall $\Theta(n \log n)$ preprocessing time.

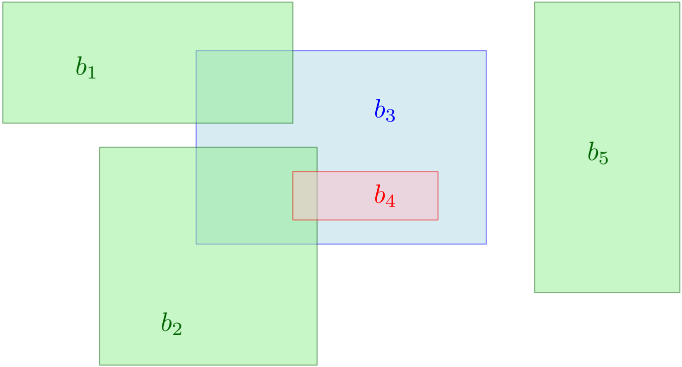
Rectangle Intersection

Problem (rectangle intersection reporting)

Given a set a set B of n boxes in $\mathbb{R}^2$, report all the pairs of boxes $b, b' \in B$ such that $b \cap b' \neq \emptyset$.

- Using segment trees, we give an $O(k + n \log n)$ time algorithm when k is the number of intersecting pairs.
  - This is optimal.
  - Faster than our line segment intersection algorithm.
- Space usage: $\Theta(n \log n)$ due to segment trees.
  - Space usage is not optimal.
    (Space $O(n)$ is possible with the same running time.)

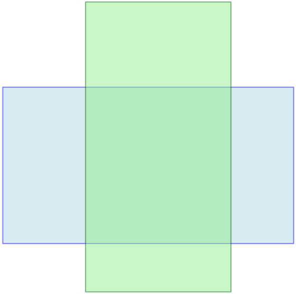
Example



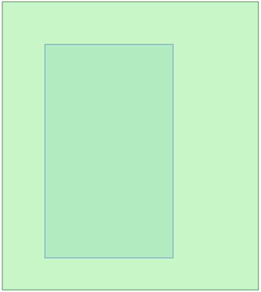
OUTPUT: (b_1, b_3),(b_2, b_3),(b_2, b_4),(b_3, b_4).

Two Types of Intersections

Overlap



Inclusion



- Intersecting edges.
  - Reduces to intersection reporting for axis-aligned segments.

- We can find them using stabbing queries.

Reporting overlaps

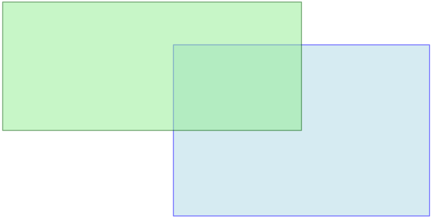
- Equivalent to reporting intersecting edges.
- Plane sweep approach.
- Sweep line status: BBST containing the horizontal line segments that intersect the sweep line, by increasing y-coordinates.
- Each time a vertical line segment is encountered, report intersection by range searching in the BBST.
- Preprocessing time: $O(n \log n)$ for sorting endpoints.
- Running time: $O(k + n \log n)$.

Reporting inclusions

- Still using plane sweep.
- Sweep line status: the boxes that intersect the sweep line ℓ, in a segment tree with respect to y-coordinates.
  - The endpoints are the y-coordinates of the horizontal edges of the boxes.
  - At a given time, only rectangles that intersect ℓ are in the segment tree.
  - We can perform insertion and deletions in a segment tree in $O(\log n)$ time.
- Each time a vertex of a box is encountered, perform a stabbing query in the segment tree.

Remarks

- At each step a box intersection can be reported several times.
- In addition there can be overlap and vertex stabbing a box at the same time.



- To obtain each intersecting pair only once, make some simple checks. (How?)

Stabbing Queries in Higher Dimension

Problem (Stabbing queries in $\mathbb{R}^d$)

Preprocess a set B of n boxes in $\mathbb{R}^d$, so as to be able to report quickly all the boxes that contain a query point q.

Approach:

- We use a multi-level segment tree.
- Inductive definition, induction on d.
- First, we store B in a segment tree $\mathcal{T}$ with respect to x_1-coordinate.
- At each node u of $\mathcal{T}$, store a $(d-1)$-dimensional multi-level segment tree over L_u, with respect to $(x_2, x_3 \dots x_d)$.

Stabbing Queries in Higher Dimension

We assume we are in fixed dimension, that is, $d = O(1)$.

Answering queries:

- Search for q in $\mathcal{T}$.
- For each node in the search path, query recursively the $(d-1)$-dimensional multi-level segment tree.
- There are $O(\log n)$ such queries.
- By induction on d, we can prove:
  - Query time: $O(k + \log^d n)$.
  - Space usage: $O(n \log^d n)$.
  - Preprocessing time : $O(n \log^d n)$.