CSE520: Computational Geometry
Lecture 14
Voronoi Diagrams

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Introduction

- References for this lecture: Textbook Section 7.1

Notation

- The *Euclidean distance* between two points $p = (p_x, p_y)$ and $q = (q_x, q_y)$ is

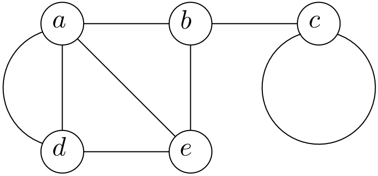
$$d(p,q) = \sqrt{(q_x - p_x)^2 + (q_y - p_y)^2}.$$

- We may also write it $|pq| = d(p,q)$.
- This is the usual distance. We will use it throughout this course, unless specified otherwise.

Notation

Definition

The *degree* of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.



- In the graph above, $\deg(a) = 4$, $\deg(b) = 3$, $\deg(c) = 3$, $\deg(d) = 3$, $\deg(e) = 3$.

The Handshaking Theorem

Theorem (Handshaking Theorem)

Let $G = (V, E)$ be an undirected graph with m edges. Then

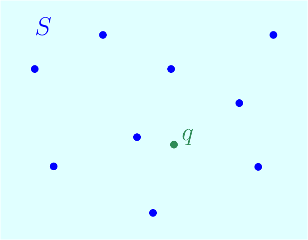
$$2m = \sum_{v \in V} \deg(v)$$

- Note that this applies even if multiple edges and loops are present.

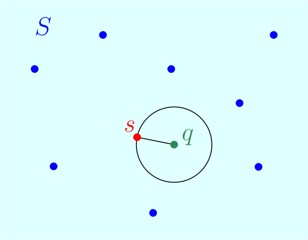
Proof.

Each loop $\{u, u\}$ contributes 2 to $\deg(u)$. Each edge $\{u, v\}$ such that $u \neq v$ contributes 1 to $\deg(u)$ and 1 to $\deg(v)$. So each edge, whether it is a loop or not, contributes 2 to the sum of the degrees. □

Nearest Neighbor Search (NNS)



Input



Output

Definition (Nearest neighbor search in $\mathbb{R}^2$)

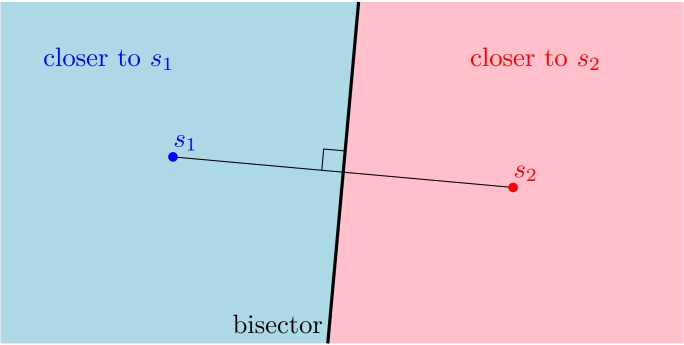
Preprocess a set S of n points in $\mathbb{R}^2$ so as to be able to answer the following queries efficiently.

- Query: point $q \in \mathbb{R}^2$.
- Output: point $s \in S$ that is closest to q.

Nearest Neighbor Search (NNS)

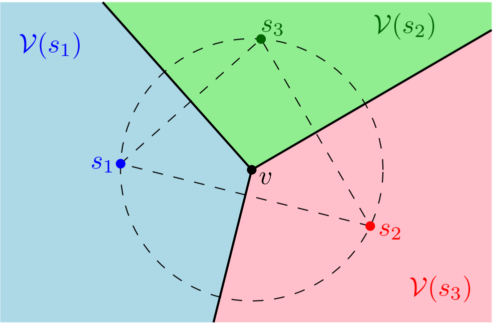
Approach:

- Draw a diagram.
- Example with $|S| = 2$:



- This is the *Voronoi diagram* of $S = \{s_1, s_2\}$.

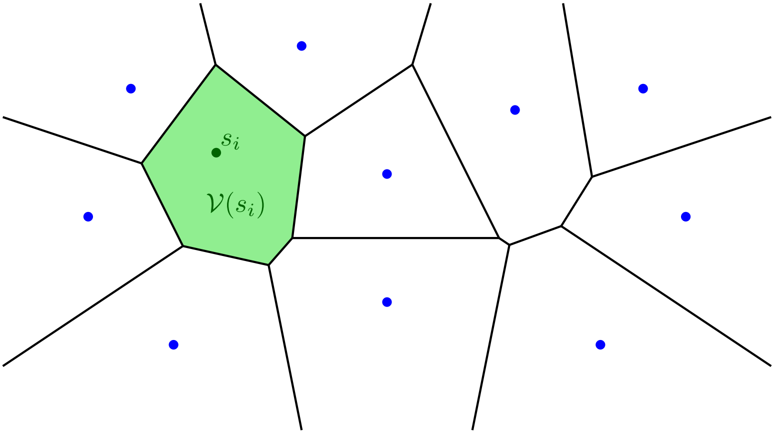
Example with $|S| = 3$



The Voronoi diagram $\mathrm{Vor}(S)$ of $S = \{s_1, s_2, s_3\}$ consists of:

- The *Voronoi vertex* v, which is the center of the circumcircle of the triangle $s_1 s_2 s_3$.
- The *Voronoi cells* $\mathcal{V}(s_1)$, $\mathcal{V}(s_2)$, $\mathcal{V}(s_3)$.
- 3 *Voronoi edges*.

Example



$$\mathcal{V}(s_i) = \{q \in \mathbb{R}^2 \mid \; d(q, s_i) < d(q, s_j) \text{ for all } j \neq i\}$$

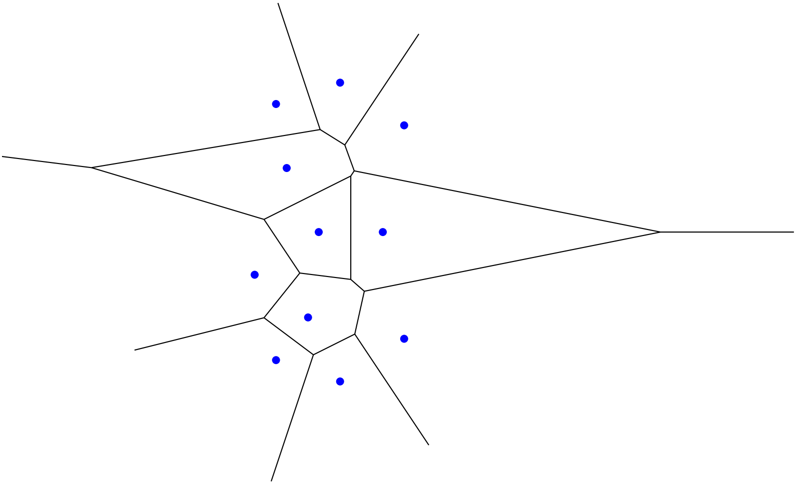
Definition

- Let S be a set of n points $\{s_1, \ldots, s_n\}$ in the plane, called *sites*.
- The Voronoi cell of s_i is the set of points whose closest point in S is s_i. More precisely,

$$\mathcal{V}(s_i) = \{q \in \mathbb{R}^2 \mid \; d(q, s_i) < d(q, s_j) \text{ for all } j \neq i\}.$$

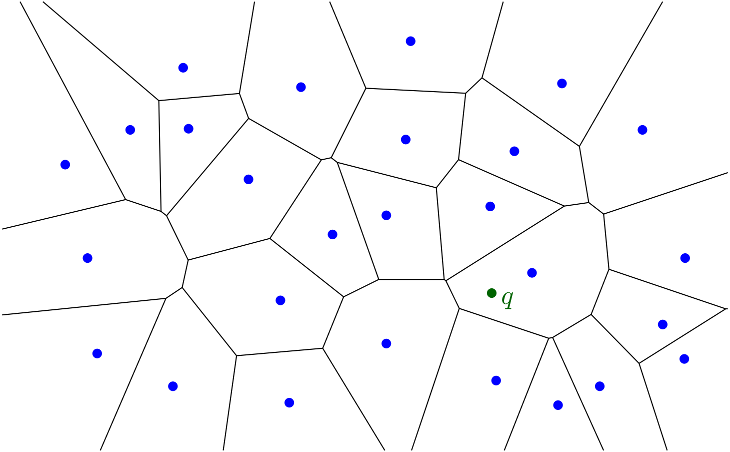
- The Voronoi diagram $\mathrm{Vor}(S)$ of S is the partition of the plane induced by the Voronoi cells.
- Its edge and vertices are called Voronoi edges and Voronoi vertices, respectively.

Example

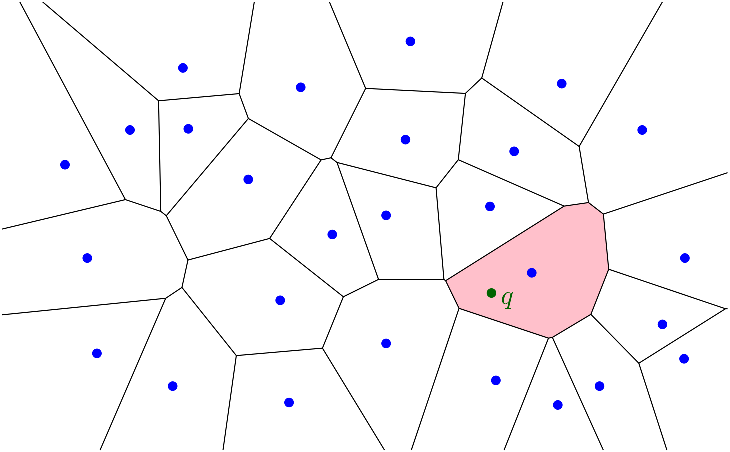


- Voronoi edges are either line segments, or (infinite) half-lines.

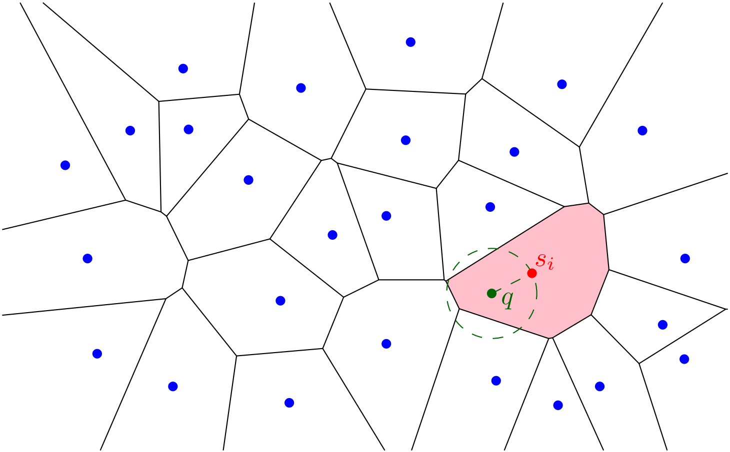
Nearest Neighbor Search (NNS)



Nearest Neighbor Search (NNS)



Nearest Neighbor Search (NNS)

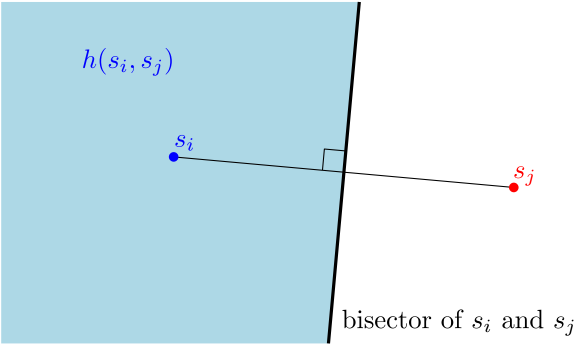


Nearest Neighbor Search (NNS)

- How to perform NNS?
- Preprocessing: compute the Voronoi diagram of S, and a point location data structure for it.
- Answering queries: perform point location for q.
- Return the site s_i such that $q \in \mathcal{V}(s_i)$.

- We will see how to do this with preprocessing time $O(n \log n)$, space usage $O(n)$ and query time $O(\log n)$. (Randomized.)
- But first let us study some properties of the Voronoi diagram.

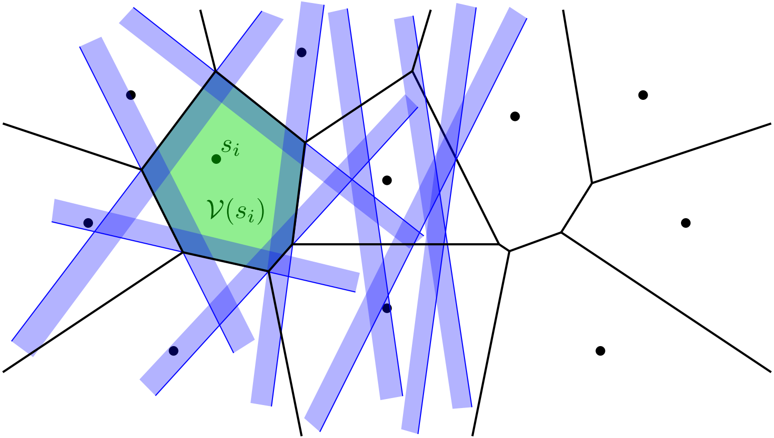
Properties



- The open halfplane bounded by the bisector of s_i and s_j, and containing s_i, is

$$h(s_i, s_j) = \{p \in \mathbb{R}^2 \mid d(p, s_i) < d(p, s_j)\}.$$

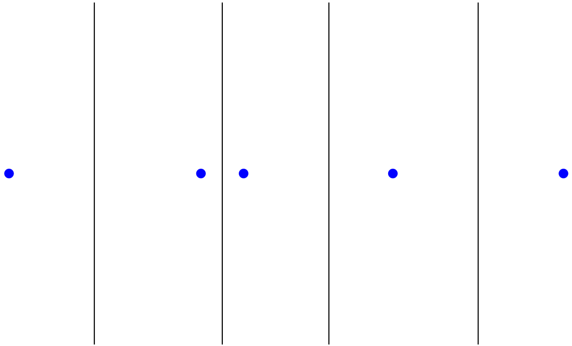
Properties



- It follows that the Voronoi cell of s_i is $\mathcal{V}(s_i) = \bigcap_{j \neq i} h(s_i, s_j)$.

Properties

- An edge of a Voronoi diagram may be a line. In which case?



Properties

Theorem (7.2)

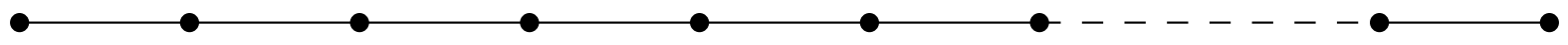
If the points in S are collinear, then the edges of $\mathrm{Vor}(S)$ are $n-1$ parallel lines. Otherwise, $\mathrm{Vor}(S)$ is connected, and its edges are either line segments or half-lines.

Properties

Theorem

The Voronoi diagram of a set of $n \geqslant 3$ sites has at most $2n - 5$ vertices and $3n - 6$ edges.

- We now want to prove this theorem.
- First attempt: The Voronoi diagram has n faces, therefore it has $O(n)$ edges and vertices.
- This argument is wrong because the graph below has 1 face, but arbitrarily many edges and vertices.



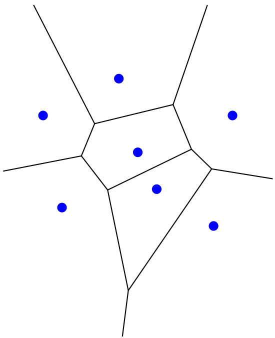
Properties

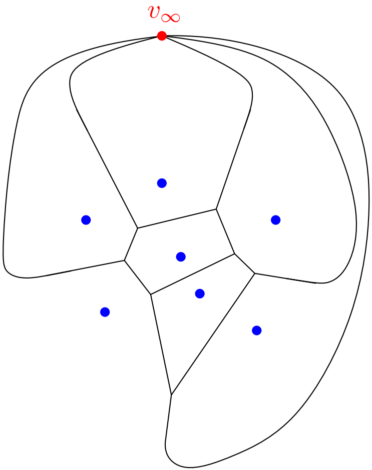
- Second attempt: Let's use Euler's formula:

$$n_v - n_e + n = 2$$

  where n_v, n_e and n are respectively the number of vertices, edges and faces of the Voronoi diagram.
- Problem: the Voronoi diagram has infinite edges, so strictly speaking, it is not a planar graph.
- How to fix it?
- Connect all infinite edges to a new vertex v_∞. (See next slide.)

Properties



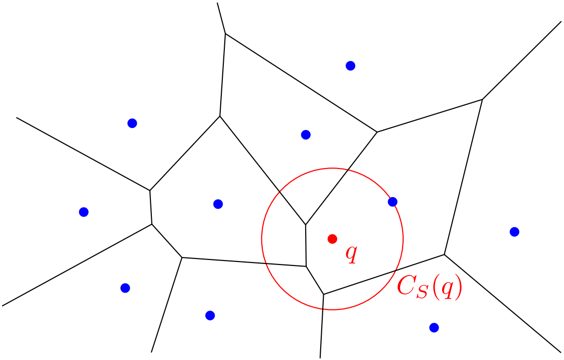


- The new graph has n faces, n_e edges and $n_v + 1$ vertices.

Properties

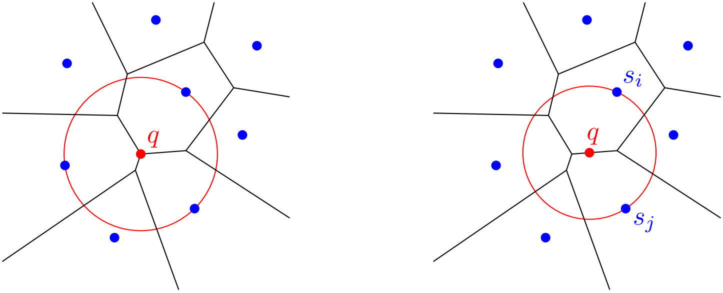
- It follows that
$$(n_v + 1) - n_e + n = 2 \tag{1}$$
- By the handshaking theorem,
$$\sum_{v \in S \cup \{v_\infty\}} \deg(v) = 2n_e.$$
- In addition, each vertex has degree at least 3, so $3(n_v + 1) \leqslant 2n_e$.
- Combined with (1), we obtain $\frac{2}{3}n_e - n_e + n \geqslant 2$, and thus $n_e \leqslant 3n - 6$.
- It follows that $n_v \leqslant 2n - 5$, which completes the proof.

Properties



- The *largest empty circle* of a point q with respect to S is the largest circle $C_S(q)$ centered at q that does not contain any site in its interior.

Properties



(a) (b)

Theorem (7.4)

The following hold for the Voronoi diagram $\mathrm{Vor}(S)$ of a set of sites S.

(a) *A point q is a Voronoi vertex iff there are at least 3 sites on $C_S(q)$.*

(b) *The bisector of s_i and s_j defines a Voronoi edge iff there is a point q on this bisector such that $C_S(q) \cap S = \{s_i, s_j\}$.*