CSE520 Computational Geometry
Lecture 15
Delaunay Triangulations

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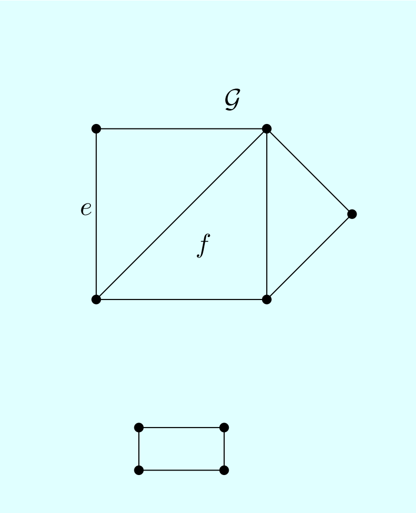
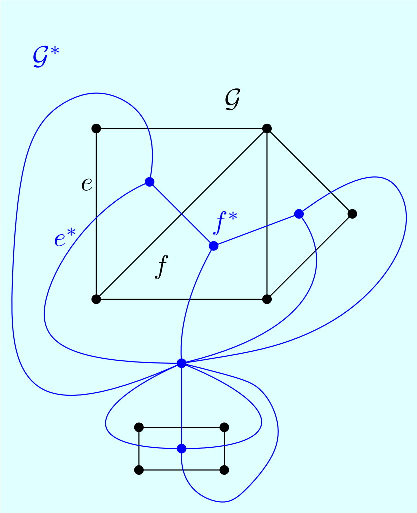
Introduction

- References for this lecture: Textbook Chapter 9.

Dual of a Plane Graph





Dual of a Plane Graph

Definition

Let $\mathcal{G}$ be a *plane graph*, that is, a planar graph embedded in the plane.
The *dual graph* $\mathcal{G}^*$ is such that:

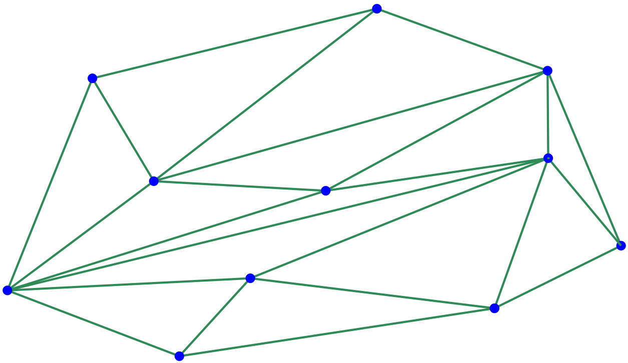
- For each face f of $\mathcal{G}$, there is a vertex f^* in $\mathcal{G}^*$
- For each edge e of $\mathcal{G}$ separating faces f and g, there is an edge $e^* = (f^*, g^*)$ in $\mathcal{G}^*$.

The vertex v^* is called the *dual vertex* of f, and e^* is the *dual edge* of e.

Property

The dual of a plane graph is planar.

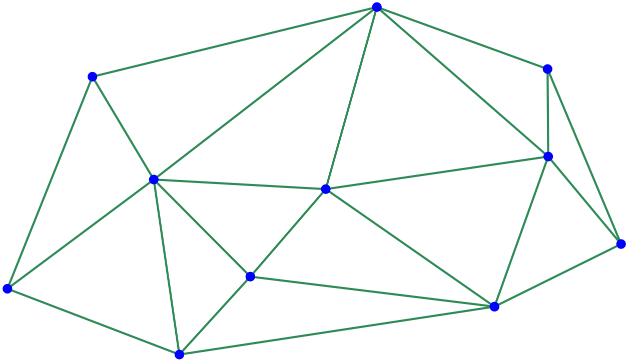
Triangulation of a Point-Set



Definition (Point-set triangulation)

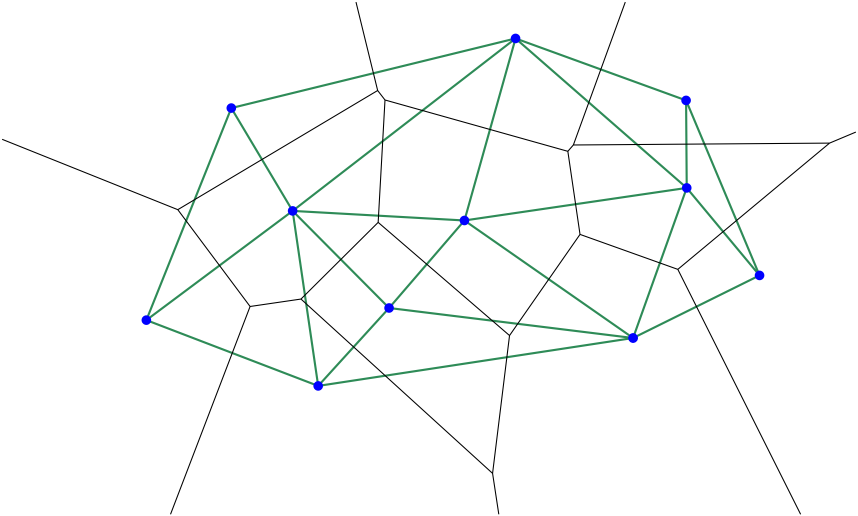
Given a set S of n points in $\mathbb{R}^2$, a *triangulation* of S is a planar graph with vertex set S, such that all the bounded faces are triangles, and these faces form a partition of the convex hull $\mathcal{CH}(S)$ of S.

The Delaunay Triangulation



- The *Delaunay triangulation* of the same set.
- It has many useful properties.

The Delaunay Triangulation



The Delaunay Triangulation

Definition (Delaunay triangulation)

Let S be a set of n points in $\mathbb{R}^2$. We assume general position in the sense that no 4 points in S are cocircular. The *Delaunay triangulation* $\mathcal{DT}(S)$ of S is the dual graph of the Voronoi diagram of S such that:

- Each vertex $\mathcal{V}(s_i)^*$ is located at the corresponding site s_i.
- The edges of $\mathcal{DT}(S)$ are straight line segments.

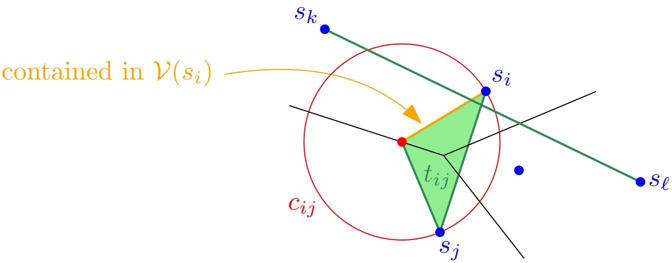
The Delaunay Triangulation

- Is $\mathcal{DT}(S)$ well defined?
- In other words, is it a triangulation?
- We need to prove that:
- Edges do not intersect (so it is a PSLG),
- and faces are triangles.

- The number of edges in a face of $\mathcal{DT}(S)$ is the degree of the corresponding Voronoi vertex.
- Our general position assumption implies that Voronoi vertices have degree 3, so faces are indeed triangles.

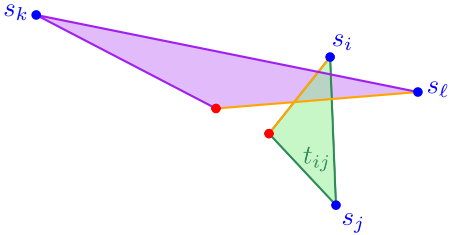
- We still need to prove that edges do not intersect.

Proof



- Suppose $\overline{s_i s_j}$ intersects $\overline{s_k s_\ell}$.
- Let C_{ij} be a circle through s_i and s_j and centered at the Voronoi edge between s_i and s_j. Let t_{ij} be the triangle formed by s_i, s_j and the center of C_{ij}.
- s_k and s_ℓ are outside C_{ij}, so $\overline{s_k s_\ell}$ must intersect an edge of t_{ij} other than $\overline{p_i p_j}$.

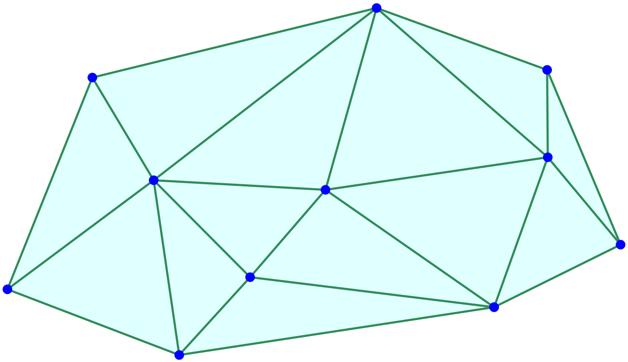
Proof



- Similarly, $\overline{p_i p_j}$ must intersect two edges of t_{kl}.
- So the boundaries of t_{ij} and $t_{k\ell}$ intersect at least three times.
- As they must cross an even number of times, they intersect at least 4 times.
- So one edge incident to the center of C_{ij} crosses an edge incident to the center of $C_{k\ell}$.
- This is impossible as these edges are contained in different Voronoi cells.

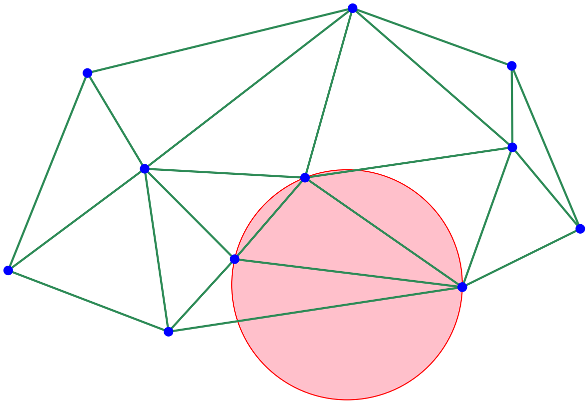
Convex Hull

- The convex hull of S is the complement of the unbounded face of $\mathcal{DT}(S)$.



- Consequence: It takes time $\Omega(n \log n)$ to compute the Delaunay triangulation.

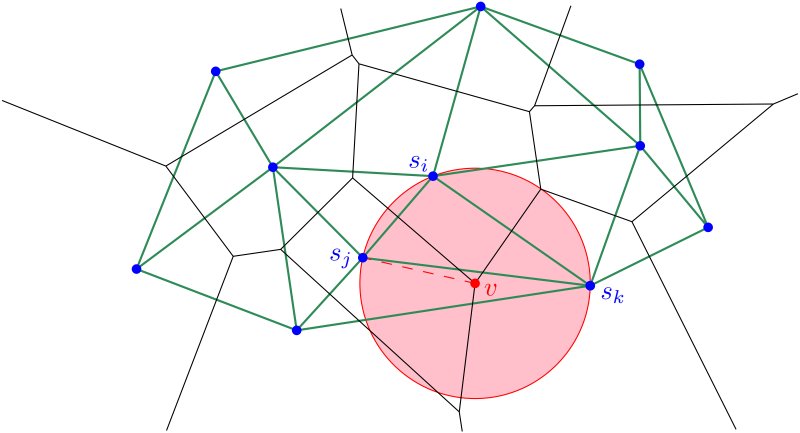
Circumcircle Property



Property (Circumcircle)

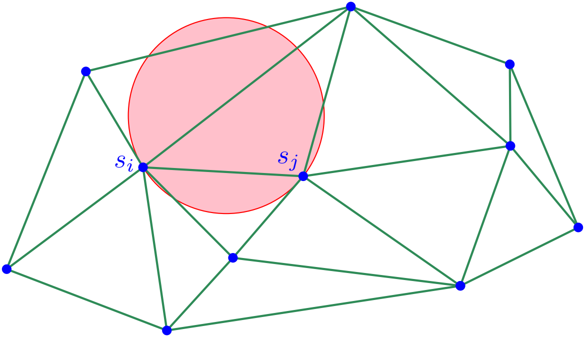
The circumcircle of any triangle in $\mathcal{DT}(S)$ is empty: It contains no site s_i in its interior.

Circumcircle Property



- Proof: Let $s_i s_j s_k$ be a triangle in $\mathcal{DT}(S)$, let v be the corresponding Voronoi vertex. Last theorem in previous lecture: the circle centered at v through $s_i s_j s_k$ is empty.

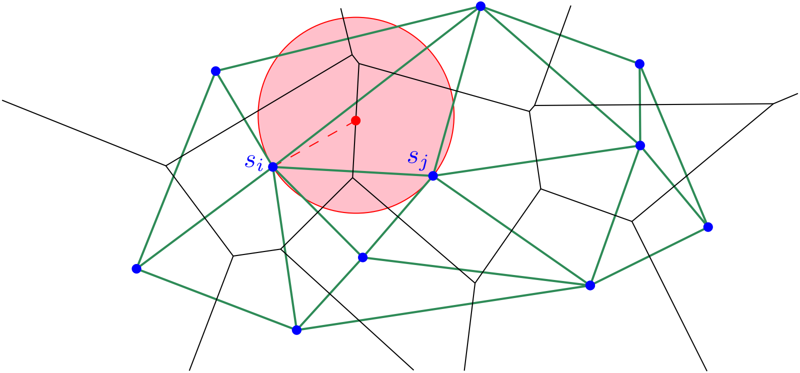
Empty Circle Property



Property (Empty circle)

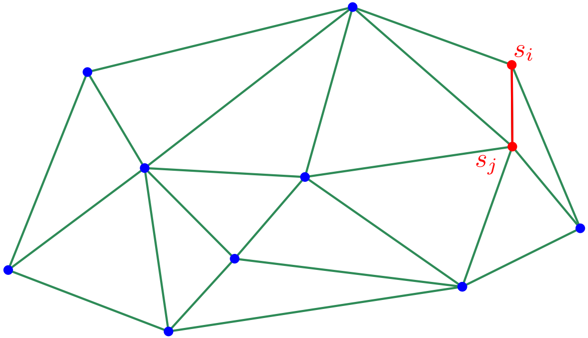
$\overline{s_i s_j}$ is an edge of $\mathcal{DT}(S)$ iff there is an empty circle through s_i and s_j.

Proof (Empty Circle Property)



- Proof: Follows from last theorem in previous lecture.

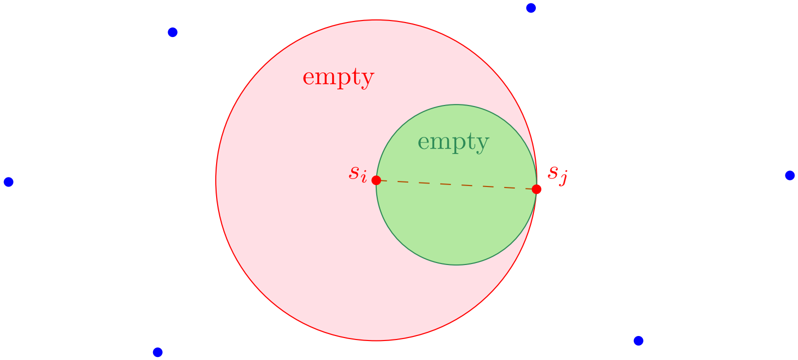
Closest Pair Property



Property (Closest pair)

The closest two sites s_i and s_j are connected by an edge of $\mathcal{DT}(S)$.

Proof (Closest Pair Property)

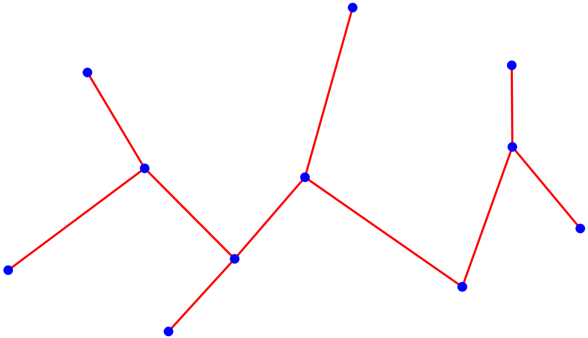


Closest Pair Property

- So we can compute the closest pair of a set of points in the plane in $O(n \log n)$ time: First compute $\mathcal{DT}(S)$, then report the shortest edge.

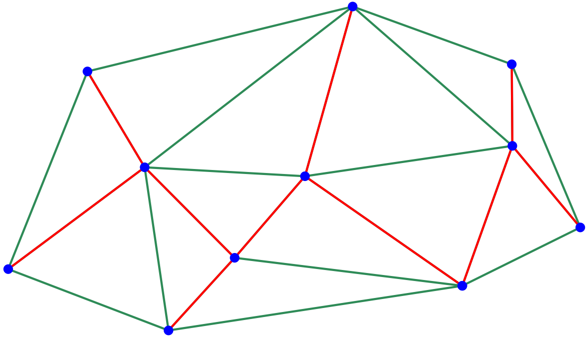
- There is another $O(n \log n)$ time algorithm presented in the undergraduate algorithms course CS E331.
- It is a divide and conquer algorithm.
- It can be generalized to arbitrary *fixed* dimension, i.e. in $\mathbb{R}^d$ with $d = O(1)$.

Euclidean Minimum Spanning Tree



- The *Euclidean graph* of a set of points S has vertex set S, and an edge of weight $d(u, v)$ between any two vertices u and v.
- The *Euclidean Minimum Spanning Tree* (EMST) is its minimum spanning tree. In other words, it is the tree of minimum length over S.

Euclidean Minimum Spanning Tree

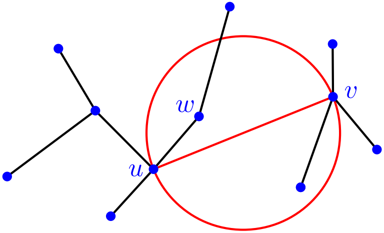


Property

The EMST is a subgraph of $\mathcal{DT}(S)$

Euclidean Minimum Spanning Tree

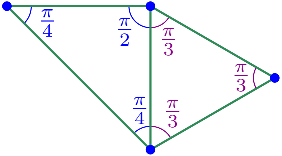
- Proof: (See Dave Mount's lecture notes.)



- Corollary: The EMST can be computed in $O(n \log n)$ time.

Angle Sequence

- Let $\mathcal{T}$ be a triangulation of S.
- Angle sequence $\Theta(\mathcal{T})$: Sequence of all the angles of the triangles of $\mathcal{T}$ in non-decreasing order.
- Example:



$$\Theta(\mathcal{T}) = (\pi/4, \pi/4, \pi/3, \pi/3, \pi/3, \pi/2)$$

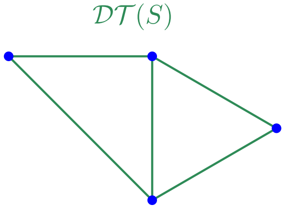
- Comparison: Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulations of S. We compare $\Theta(\mathcal{T})$ and $\Theta(\mathcal{T}')$ using *lexicographic order*.
- Example: $(1,1,3,4,5) < (1,2,4,4,4)$.

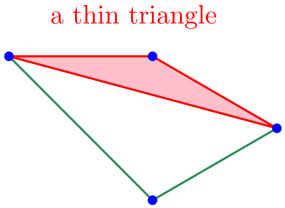
Optimality of the Delaunay Triangulation $\mathcal{DT}(S)$

Theorem

Let S be a set of points in general position. Then the angle sequence of $\mathcal{DT}(S)$ is maximal among all triangulations of S.

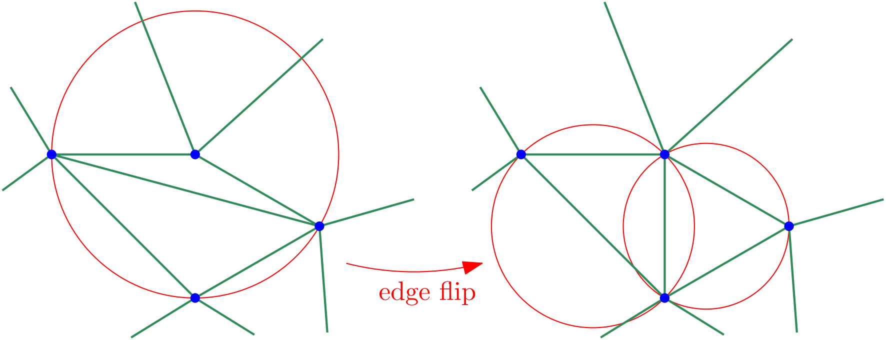
- So the Delaunay triangulation maximizes the minimum angle.
- Intuition: Avoids thin triangles



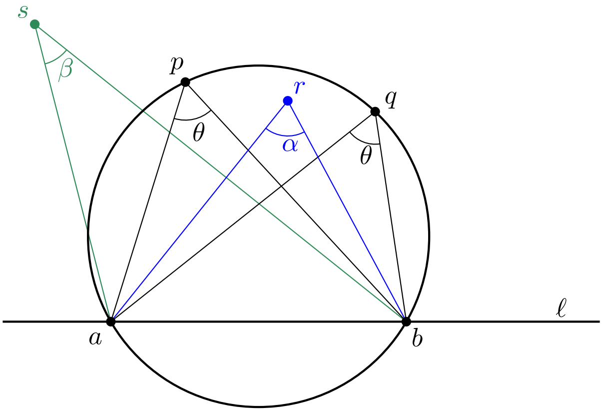


Proof of Optimality

- Flip edges in order to enforce the circumcircle property.
- It increases the angle sequence.

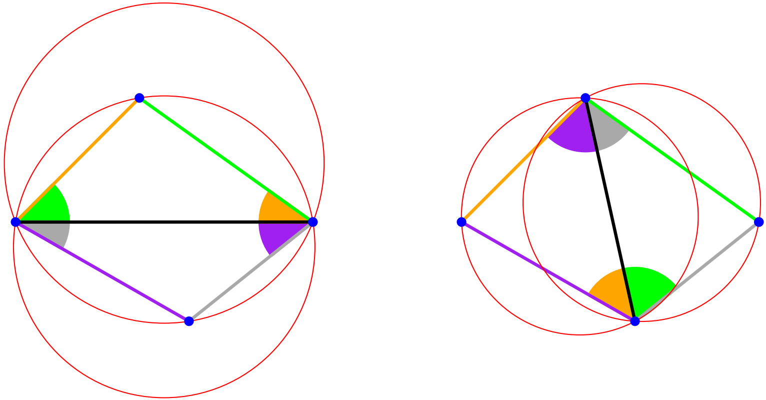


Proof of Optimality



- Based on this theorem (Thales): If r is inside the circle, s is outside, and p, q, r, s are on the same side of ℓ, then $\beta < \theta < \alpha$, where $\alpha = \angle arb$, $\theta = \angle apb = \angle aqb$ and $\beta = \angle asb$

Proof of Optimality

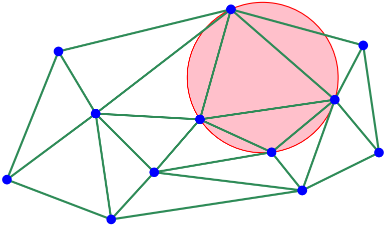


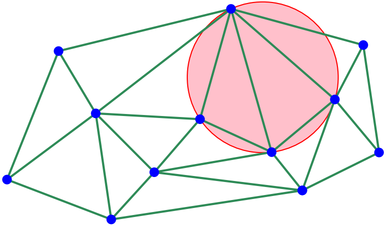
- The green angle on the left is smaller than the green angle on the right. Same with other colors.

(See Dave Mount's notes.)

Degenerate Cases

- In degenerate cases, there may be several Delaunay triangulations.
- Example with two possibilities:





- Any Delaunay triangulation maximizes the minimum angle. But the angle sequences of two Delaunay triangulations may differ.