CSE520: Computational Geometry
Lecture 18
Point-Line Duality

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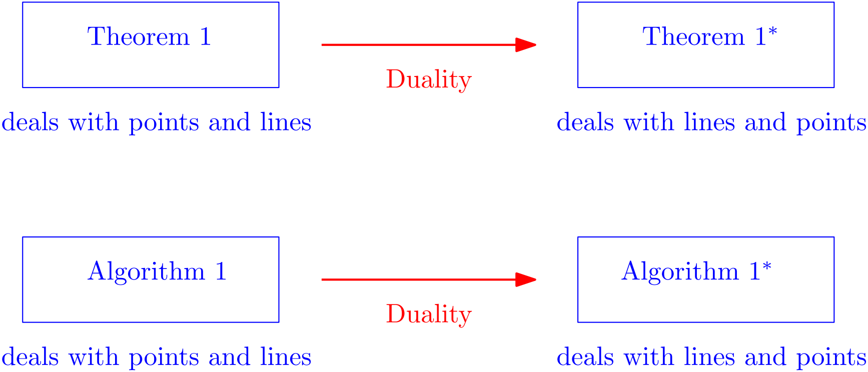
Outline

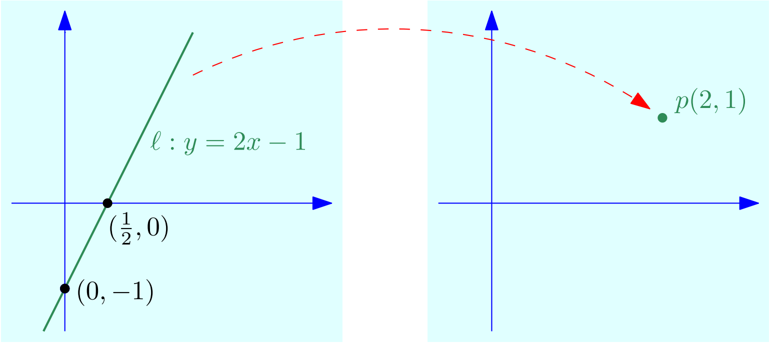
- We saw planar graph duality in Lecture 10.
- In this lecture, a different type of duality: point-line duality.
- Reference: Textbook Chapter 8.

Point-Line Duality

- Point-Line Duality: A transformation that exchanges points and lines.
- Motivation:



Point-Line Duality: Example



- notation:

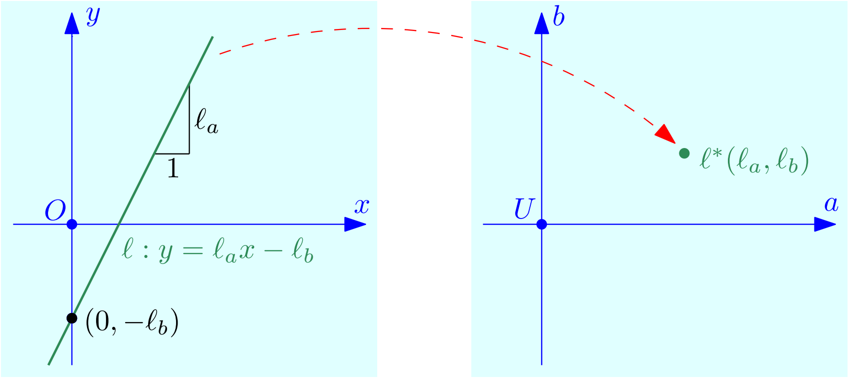
$$p = \ell^*$$

Dual Point

- Let ℓ be a non-vertical line.
- ℓ has equation $\ell \ : \ y = \ell_a x - \ell_b$.
- ℓ_a is the *slope* of ℓ.
- We associate the point $\ell^* = (\ell_a, \ell_b)$ to ℓ.
- ℓ^* is called the *dual* of ℓ.

- We say that ℓ is a line in the *primal plane*.
- Here, the primal plane is associated with a coordinate frame Oxy.
- We say that ℓ^* is a point in the *dual plane*.
- We will use a coordinate frame Uab for the dual plane.

Dual Point



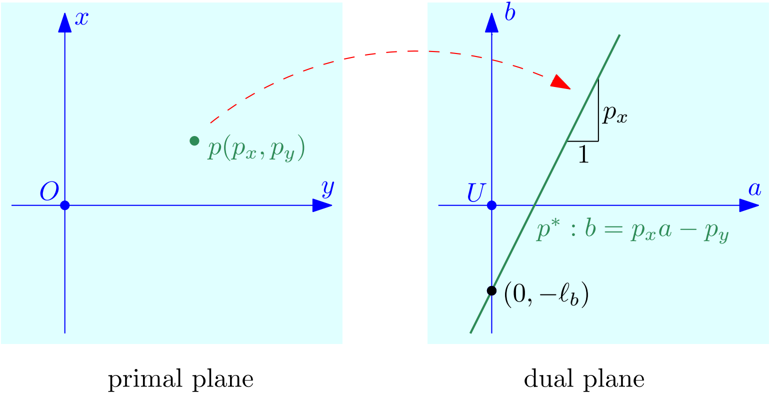
primal plane

dual plane

Dual Line

- Let $p = (p_x, p_y)$ be a point in the primal plane.
- Its dual is a line p^* in the dual plane with equation

$$p^* \ : \ b = p_x a - p_y.$$



Self Inverse

Property (Self inverse)

For any point p in the primal plane, $(p^*)^* = p$.

Proof.

Let $p = (p_x, p_y)$ be a point in the primal plane. Then its dual is the line $p^* \ : \ b = p_x a - p_y$. It follows that $(p^*)^* = (p_x, -(-p_y)) = p$. □

Property (Self inverse)

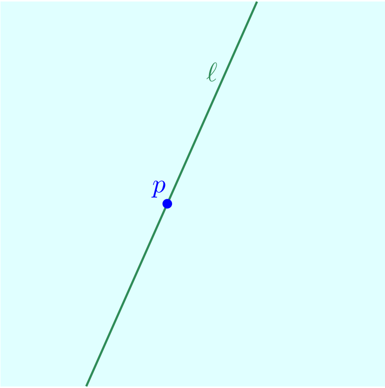
For any line ℓ of the primal plane, $(\ell^*)^* = \ell$.

Proof.

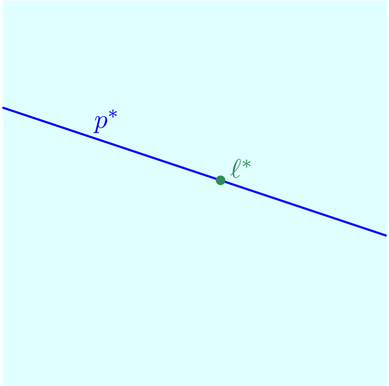
- $\ell \ : \ y = \ell_a x - \ell_b$.
- $\ell^* = (\ell_a, \ell_b)$.
- $(\ell^*)^* \ : \ y = \ell_a x - \ell_b$.

□

Point-Line Duality is Incidence Preserving



primal plane



dual plane

Point-Line Duality is Incidence Preserving

Property

$p \in \ell$ *iff* $\ell^* \in p^*$.

Proof.

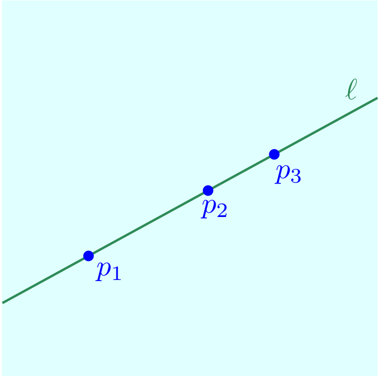
$$
\begin{aligned}
p \in \ell &\Leftrightarrow p_y = \ell_a p_x - \ell_b && \text{because } \ell : y = \ell_a x + b \\
&\Leftrightarrow \ell_b = p_x \ell_a - p_y \\
&\Leftrightarrow (\ell_a, \ell_b) \in p^* && \text{because } p^* : b = p_x a - p_y \\
&\Leftrightarrow \ell^* \in p^* && \text{because } \ell^* = (\ell_a, \ell_b)
\end{aligned}
$$

□

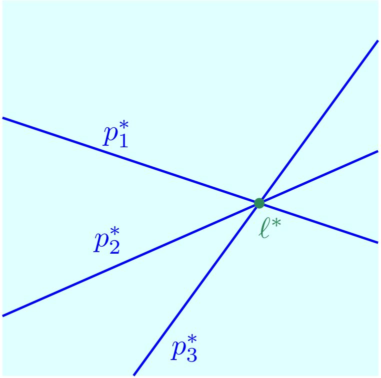
Multiple Incidence

Corollary

p_1, p_2 and p_3 are collinear iff p_1^*, p_2^* and p_3^* intersect at a common point.



primal plane

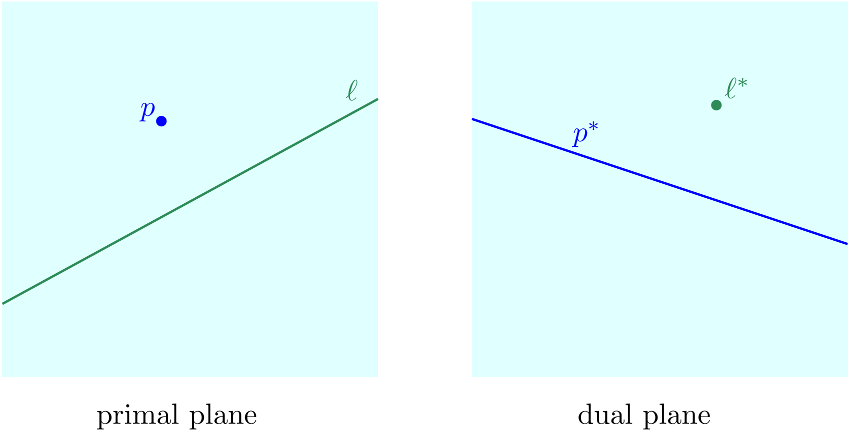


dual plane

Order Reversing

Property (Order reversing)

p lies above ℓ iff ℓ^* is above p^*.



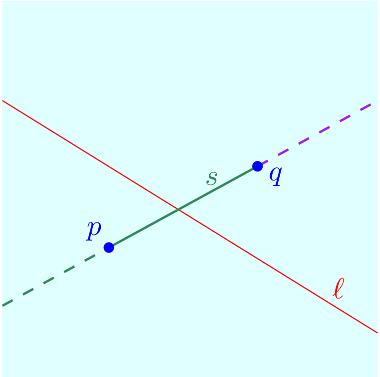
Example: Dual of a Line Segment

- Let $s = \overline{pq}$ be a line segment in $\mathbb{R}^2$.
- How can we define its dual?
- Its dual s^* is the union of the dual lines of the points of s.
- All the points in s are collinear, so all the lines in s^* pass through one point.
- So it is a double wedge. (See next slide.)

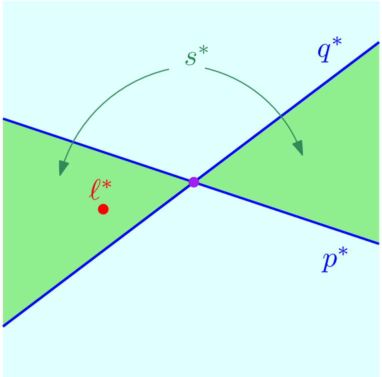
Property

A line ℓ intersects a segment s iff ℓ^* is in s^*.

Example: Dual of a Line Segment

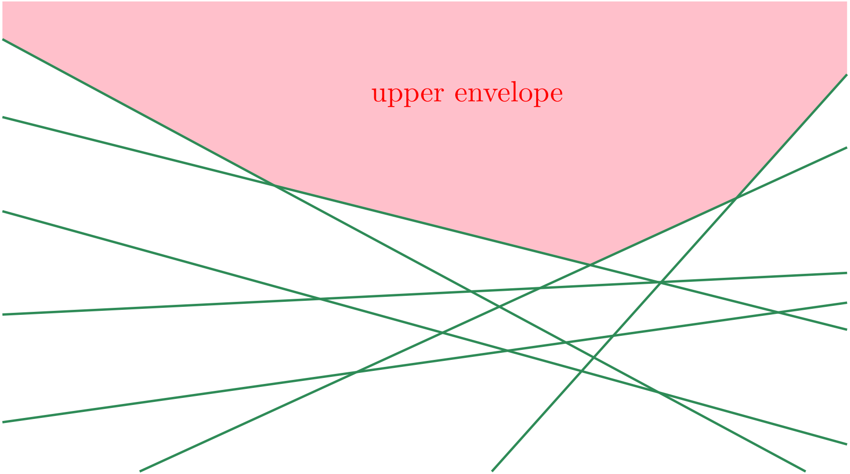


primal plane



dual plane

Upper Envelope of Lines



Upper Envelope of Lines

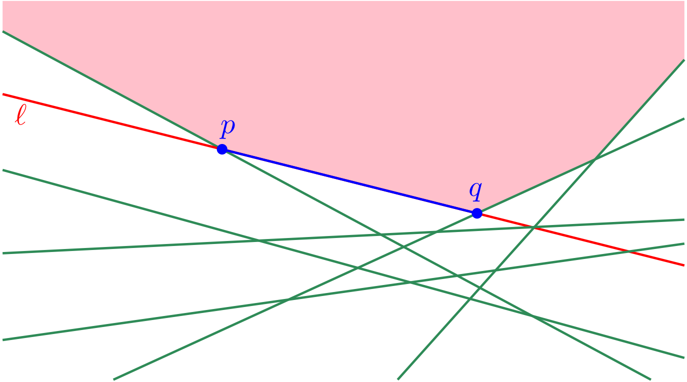
Definition (Upper envelope)

The upper envelope of a set of lines is the set of the points that are above all lines.

- How to compute the upper envelope of a set L of n lines?
- We will use duality.
- configuration of lines $\Leftrightarrow$ configuration of points
- We denote $L^* = \{\ell^* \mid \ell \in L\}$.

Observation

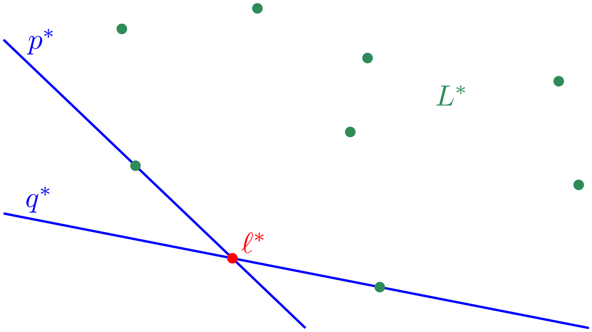
- Suppose that ℓ appears as a segment $\overline{pq}$ in the upper envelope.



- Interpretation in the dual space?

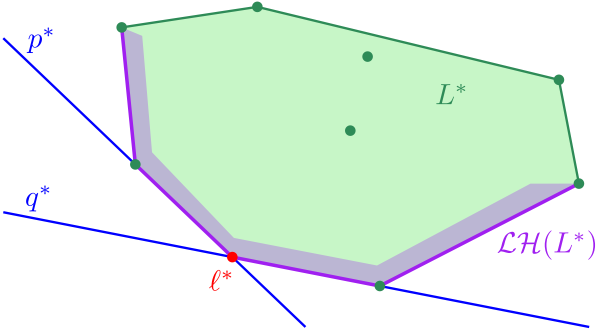
Observation

- p and q are on or above all the lines in L.
- So p^* and q^* are on or below all the points in L^*.



Observation

- So p^* and q^* are on the lower hull $\mathcal{LH}(L^*)$ of L^*.
- $\ell^* = p^* \cap q^*$ is also on $\mathcal{LH}(L^*)$



Consequences

- So the lines that appear in the upper envelope of L correspond to the vertices of the lower hull of L^*.
- How to compute the upper envelope?
- Compute the lower hull $\mathcal{LH}(L^*)$.
- Traverse this chain from left to right, output the dual of the vertices.
- This gives you a list of the lines in L.
- These are the lines that appear in the upper envelope.
- These lines are in the same order as they appear in this upper envelope, from left to right.
- Why?

Consequences

- We can compute the lower hull using the algorithm from Lecture 2.
- It takes $O(n \log n)$ time.
- So we can compute an upper envelope of lines in $O(n \log n)$ time.
- So we can compute the intersection of n *upward* halfplanes (i.e. with equation $y \geqslant ax + b$) in $O(n \log n)$ time.

- To compute an arbitrary intersection of halfplanes, split them into two sets: those that go upward and those that go downward.
- The intersection of the upward halfplanes is an upper envelope of lines.
- The other subset is a lower envelope. (Similar idea.)
- Intersect these two chains. (Plane sweep for instance.)
- Overall, it takes $O(n \log n)$ time.

Remarks

- We have just seen that, in the plane, the following three problems are equivalent:
  - Convex hull of a point set.
  - Upper (lower) envelope of lines.
  - Halfspace intersection.
- In higher dimension, it is similar.
  - But the intersection of n half-spaces is a polytope that can have $\Omega(n^{\lfloor d/2 \rfloor})$ vertices.
  - Voronoi diagrams and Delaunay triangulations can be seen as upper envelopes in one dimension higher.

Exercise

Exercise (textbook 8.7)

Let R be a set of n red points in the plane, and let B be a set of n blue points in the plane. We call a line ℓ a separator for R and B if ℓ has all points of R to one side and all points of B to the other side. Give a randomized algorithm that can decide in $O(n)$ expected time whether R and B have a separator.

Exercise

Exercise (textbook 8.8)

The dual transform of Section 8.2 has minus signs. Suppose we change them to plus signs, so the dual of a point (p_x, p_y) is the line $b = p_x a + p_y$, and the dual of the line $y = \ell_a x + \ell_b$ is the point (ℓ_a, ℓ_b). Is this dual transform incidence preserving? Is it order reversing?