CSE520: Computational Geometry
Lecture 19
Arrangements of Lines

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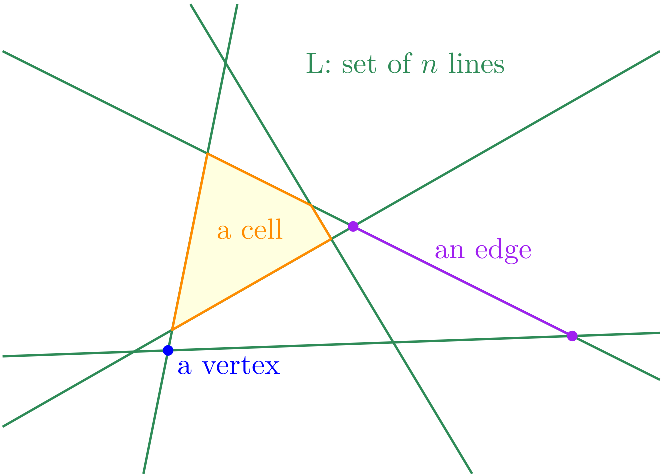
Outline

- Reference: Textbook Chapter 8.

Arrangements of Lines



arrangement $\mathcal{A}(L)$

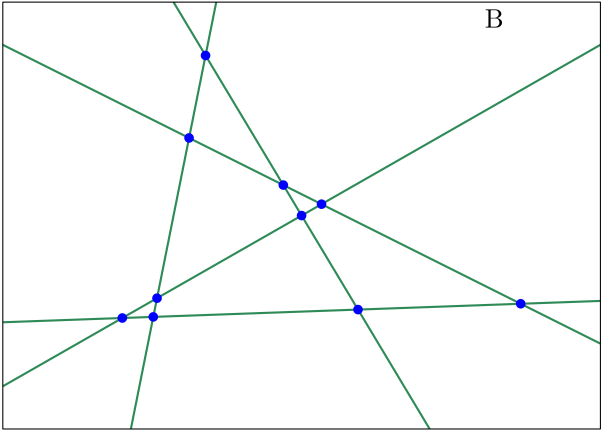
Definition

Definition (Arrangement of lines)

Let L be a set of n lines in $\mathbb{R}^2$. These lines subdivide $\mathbb{R}^2$ into several regions, called *cells*. The edges of this subdivision are line segments or half-lines. The vertices are intersection points between two lines of L. This subdivision, with adjacency relation between vertices, edges and cells, is called the arrangement $\mathcal{A}(L)$ of L.

Bounding Box

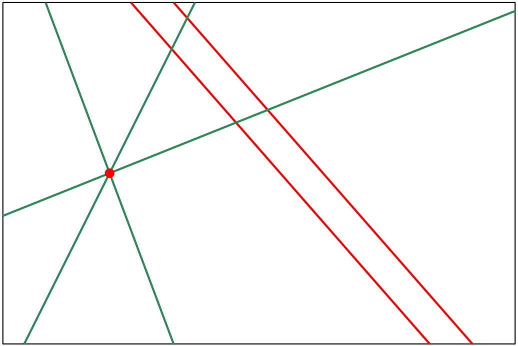
- We restrict our attention to a bounding box B that contains all the vertices of $\mathcal{A}(L)$.



- Now all edges and faces are bounded.
- How fast can we compute such a bounding box?

General Position Assumptions

- No two lines are parallel.
- No three lines intersect at one point.



A degenerate case: two lines are parallel, and three lines meet at one point.

Combinatorial Complexity

Definition (Combinatorial complexity)

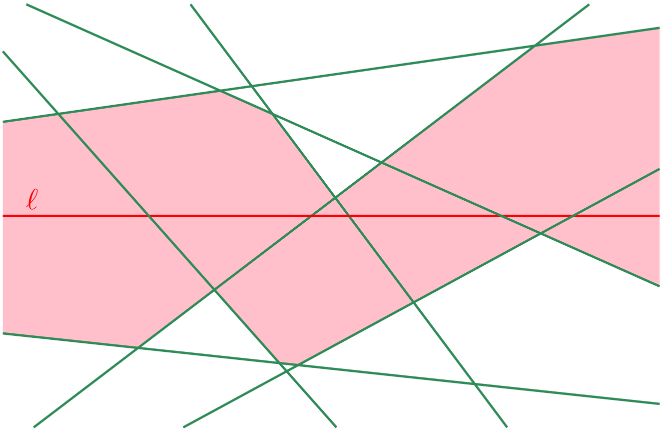
The combinatorial complexity of an arrangement $\mathcal{A}(L)$ is the total number of vertices, edges, and faces in $\mathcal{A}(L)$.

- This quantity is $\Theta(n^2)$ for an arrangement of n lines.
- More precisely, if L is in general position:
- $\mathcal{A}(L)$ has $\binom{n}{2}$ vertices.
- $\mathcal{A}(L)$ has n^2 edges.
- $\mathcal{A}(L)$ has $\binom{n}{2} + n + 1$ faces.
- Proof?

Zone

Definition (Zone)

The zone of a line $\ell \notin L$ is the set of cells in $\mathcal{A}(L)$ that intersect ℓ.



Zone Theorem

Theorem (Zone Theorem)

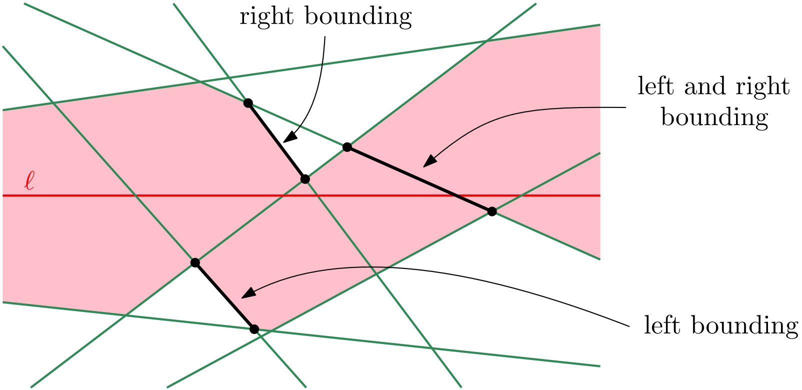
The total number of edges of all the cells in the zone of a line $\ell \notin L$ in $\mathcal{A}(L)$ is $O(n)$.

- In other words: the combinatorial complexity of the zone of a line is linear.
- Why is it not obvious?
- Some lines of L appear in several cells of the zone; See previous slide.

Proof

Definition

Left-bounding and right-bounding edges are edges that bound a cell of the zone of L from the left and from the right, respectively.

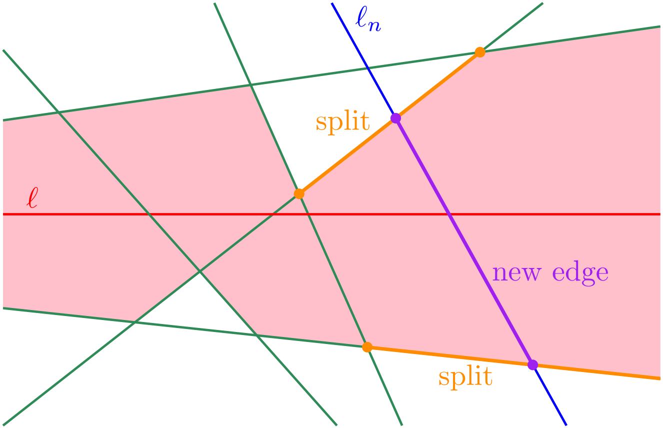


Proof

- We assume that $L \cup \ell$ is in general position.
- Without loss of generality, we assume that ℓ is horizontal.

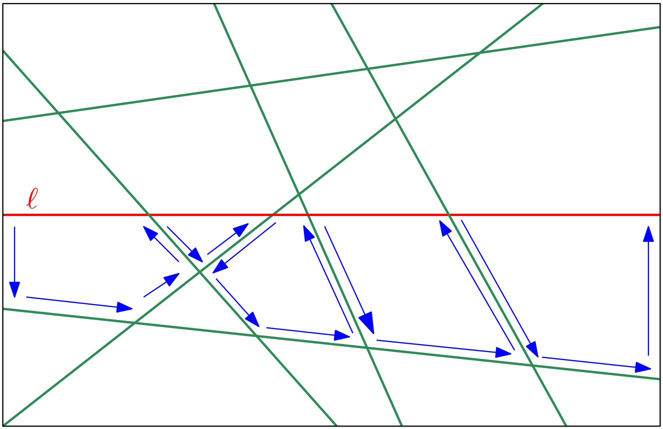
- Proof:
- We insert the lines of L from left to right, according to the x-coordinate of their intersection with ℓ.
- We will prove that inserting a line increases by at most 3 the number of left bounding edges.
- It follows that the zone has at most $3n$ left bounding edges.
- So the zone of ℓ has at most $6n$ edges.

Proof



- Inserting ℓ_n: Two left bounding edges are split and one is created.
- Total: $+3$ left bounding edges.

Constructing an Arrangement



Constructing an arrangement

Incremental algorithm:

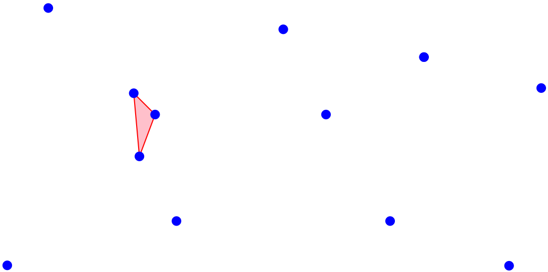
- We insert the lines one by one and update the arrangement.
- The arrangement is maintained in a Doubly Connected Edge List.

Insertion of a new line ℓ:

- Find the leftmost point of ℓ in the bounding box and the cell that contains it.
- It takes $O(n)$ time.
- Traverse the zone of ℓ from left to right and update the arrangement accordingly.
- By the Zone Theorem, it can be done in $O(n)$ time.

Overall, we compute a DCEL of $\mathcal{A}(L)$ in $O(n^2)$ time.

An Application of Arrangements and Duality



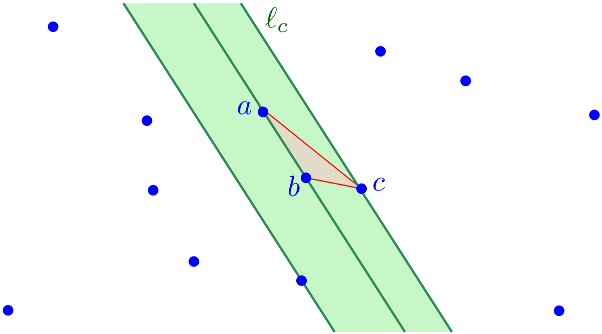
Problem (Smallest triangle)

Given a set P of n points in $\mathbb{R}^2$, find the triangle with smallest area whose vertices are in P.

- It can be solved by brute force in $O(n^3)$ time.
- We will give an $O(n^2)$ time algorithm.

Characterization

- Let $(a, b) \in P^2$
- How can we find $c \in P$ such that area of a, b, c is minimized?
- Find the largest empty corridor along line ab.



- c lies on its boundary.

Characterization

c lies on a line ℓ_c such that:

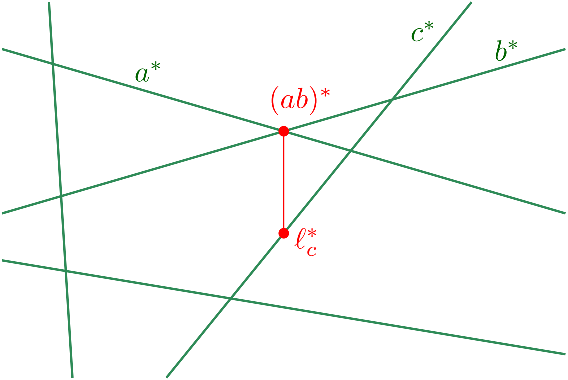
- ℓ_c is parallel to ab,
- and there is no other line with same slope between ab and ℓ_c.

What does it mean in the dual plane?

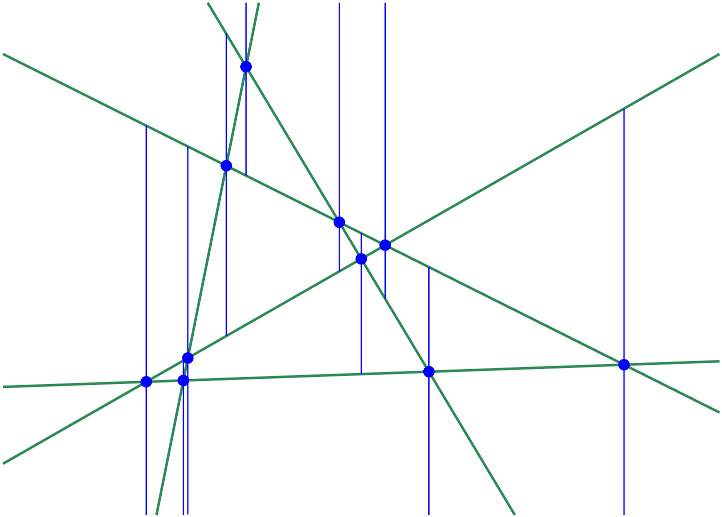
- We call $(ab)^*$ the dual of line ab.
- $(\ell_c)^*$ is on c^*.
- $(\ell_c)^*$ and $(ab)^*$ have same x-coordinate.
- No line p^* where $p \in P$ crosses the line segment with endpoints $(ab)^*$ and $(\ell_c)^*$.

(See next slide.)

Characterization



Algorithm



Algorithm

- $(\ell_c)^*$ is in the same cell of $\mathcal{A}(P^*)$ as $(ab)^*$.
- $(\ell_c)^*$ is vertically above or below $(ab)^*$.
- Once $\mathcal{A}(P^*)$ is computed, only two candidates involving a and b.
- We compute $\mathcal{A}(P^*)$ in $O(n^2)$ time.
- For each cell of this arrangement, we compute by plane sweep the point of the boundary that is vertically above or below every vertex of the cell.
- It takes time linear in the number of edges of the cell as they are given in counterclockwise order.

Algorithm

- For each triple a^*, b^*, c^* we found, we can compute the area of triangle a, b, c in $O(1)$ time.
- We maintain the minimum value found so far in $O(1)$ time by triangle we consider.
- Overall, the running time of our algorithm is proportional to the combinatorial complexity of $\mathcal{A}(P^*)$.
- So it runs in $O(n^2)$ time.