CSE331 Introduction to Algorithms
Lecture 4: Merge Sort

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Introduction

- Reference: Section 2.3 of the textbook Introduction to Algorithms by Cormen, Leiserson, Rivest and Stein.
- In Lecture 1, we presented INSERTION SORT.
- In this lecture, we present MERGE SORT, a more efficient algorithm.
- It follows a *Divide-and-Conquer* approach:
  - Sort recursively $A[1 \dots n/2]$ and $A[n/2+1 \dots n]$
  - Combine the results.
  - See next lectures for other examples of this approach.

Merging two Sorted Sequences

Problem (Merging two Sorted Sequences)

Given two sorted sequences $(a_1, a_2, \ldots, a_{n_1})$ *and* $(b_1, b_2, \ldots, b_{n_2})$*, sort the sequence* $(a_1, a_2, \ldots, a_{n_1}, b_1, b_2, \ldots, b_{n_2})$.

Example

- **INPUT:** $(1, 6, 10, 11, 18)$ and $(3, 4, 5, 15)$
- **OUTPUT:** $(1, 3, 4, 5, 6, 10, 11, 15, 18)$

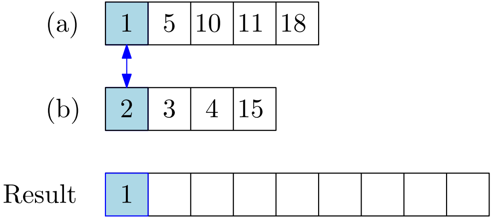
Algorithm for Merging two Sorted Sequences

(a)	1	5	10	11	18

(b)	2	3	4	15

Result									

Algorithm for Merging two Sorted Sequences



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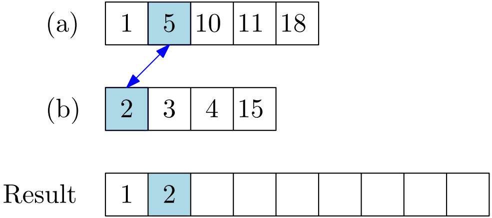

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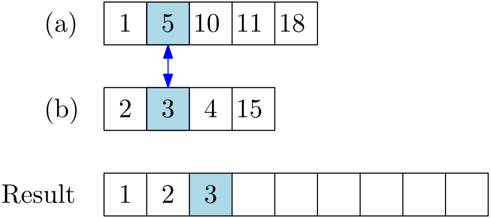

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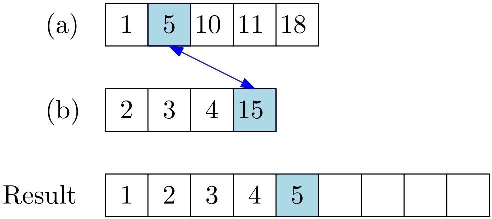
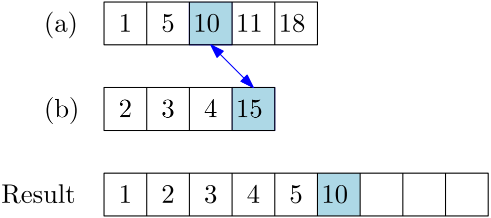
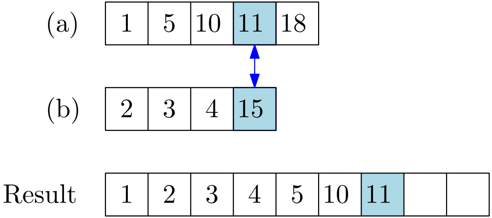


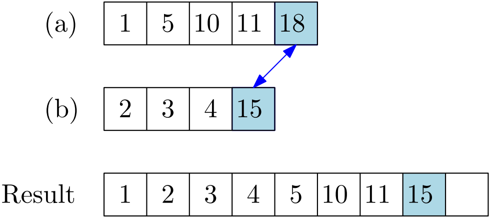
Algorithm for Merging two Sorted Sequences



Algorithm for Merging two Sorted Sequences



Algorithm for Merging two Sorted Sequences



Algorithm for Merging two Sorted Sequences

(a)	1	5	10	11	18				
(b)	2	3	4	15					
Result	1	2	3	4	5	10	11	15	18

Algorithm for Merging two Sorted Sequences

- So the algorithm works as follows:
- Keep a pointer to the current elements a_i and b_j in the input sequences, initialized to a_1 and b_1 respectively.
- At each step:
  - If $a_i \leqslant b_j$, then append a_i to the result, and move the pointer to a_i one position to the right.
  - If $a_i > b_j$, then append b_j to the result, and move the pointer to b_j one position to the right.

Algorithm for Merging two Sorted Sequences

- Next slide:
  - A more detailed pseudocode for the special case where the input sequences are two contiguous subarrays $A[p \dots q]$ and $A[q+1 \dots r]$.
  - We will place *sentinels*, represented by the value ∞, that we assume to be larger than all the keys.

	8	9	10	11	12	13	14	15	16	17
A	...	2	4	5	7	1	2	3	6	...
		k								

$p = 9$
$q = 12$
$r = 16$

	1	2	3	4	5
L	2	4	5	7	∞
	i				

	1	2	3	4	5
R	1	2	3	6	∞
	j				

Pseudocode

```
procedure MERGE(A, p, q, r)
    n1 ← q − p + 1, n2 ← r − q
    create new arrays L[1 . . . n1 + 1], R[1 . . . n2 + 1]
    for i ← 1, n1 do
        L[i] ← A[p + i − 1]
    for j ← 1, n2 do
        R[j] ← A[q + j]
    L[n1 + 1] ← ∞, R[n2 + 1] ← ∞
    i ← 1, j ← 1
    for k ← p, r do
        if L[i] ⩽ R[j] then
            A[k] ← L[i]
            i ← i + 1
        else
            A[k] ← R[j]
            j ← j + 1
```

Proof of Correctness

Loop Invariant

At the start of each iteration of the for loop of lines 10–16, the subarray $A[p \dots k-1]$ contains the $k-p$ smallest elements of $L[1 \dots n_1+1]$ and $R[1 \dots n_2+1]$ in sorted order. Moreover, $L[i]$ and $R[j]$ are the smallest elements of their arrays that have not been copied back into A.

- Proof of Initialization, Maintenance and Termination done in class. See pages 32–33 of the textbook.

Analysis

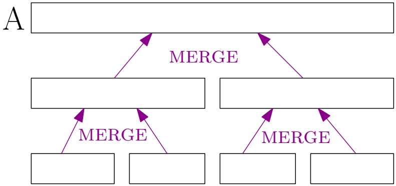
- There are respectively $n_1 + 1$, $n_2 + 1$, and $n_1 + n_2 + 1$ iterations in the three loops.
- So the running time is $c_1(n_1 + n_2) + c_2$ for some constants c_1, c_2.
- In terms of the input size $n = r - p + 1 = n_1 + n_2$, this is $c_1 n + c_2$. So we just proved:

Theorem

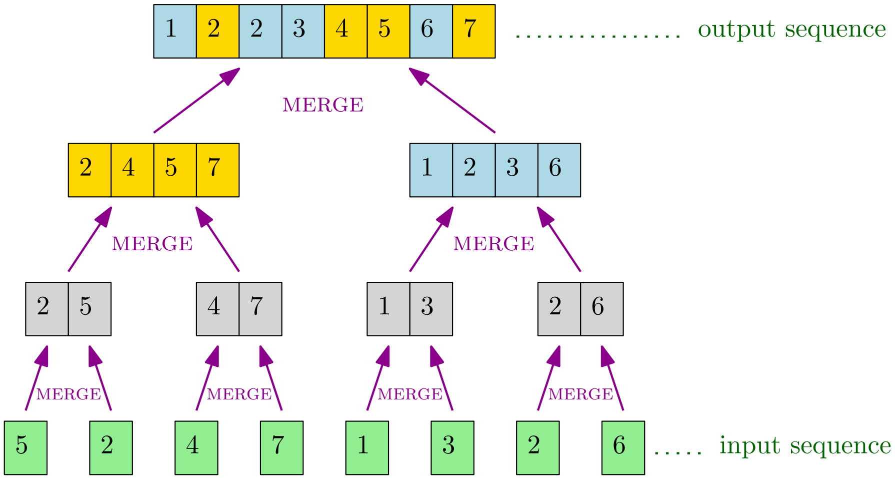
Two sorted sequences can be merged in $c_1 n + c_2$ time, where n is the sum of the lengths of the two sequences, and c_1, c_2 are two constants. In other words, two sorted sequences can be merged in linear time.

First Approach

- INSERTION SORT runs in $c_4 n^2 + c_5 n + c_6$ for some constants c_4, c_5, c_6.
- First approach:
  - (We assume $n \in 2\mathbb{N}$.)
  - Sort $A[1 \dots n/2]$ and $A[n/2+1 \dots n]$ using INSERTION SORT.
  - Merge the two results.
- Running time: $\frac{c_4}{2} n^2 + (c_1 + c_5)n + c_2 + 2c_6$
- The leading coefficient has improved by a factor 2.
- For large values of n, it is about twice faster than insertion sort.
- We can push this idea further and split A into 4 parts:
- It improves the leading coefficient further.



Merge Sort



1 2 2 3 4 5 6 7
output sequence
MERGE
2 4 5 7
1 2 3 6
MERGE
MERGE
2 5
4 7
1 3
2 6
MERGE
MERGE
MERGE
MERGE
5 2 4 7 1 3 2 6
input sequence

Merge Sort

- Merge sort splits recursively until the arrays have size 1.
  - No need to call Insertion Sort.

Pseudocode

```
procedure MergeSort(A, p, r)
    if p < r then
        q ← ⌊(p + r)/2⌋
        MergeSort(A, p, q)
        MergeSort(A, q + 1, r)
        Merge(A, p, q, r)
```

- In order to sort $A[1 \ldots n]$, call $\textsc{MergeSort}(A, 1, n)$

Analysis

- Not counting the call to MERGE and the two recursive calls, MERGE SORT takes constant time c_3.
- So the running time $T(n)$ of MERGE SORT is given by the recurrence relation:

$$T(n) = \begin{cases} c_3 & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c_1 n + c_2 + c_3 & \text{if } n \geqslant 2 \end{cases}$$

- An upper bound $U(n) \geqslant T(n)$ is given by the relation

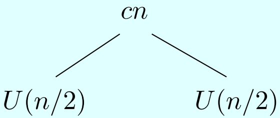
$$U(n) = \begin{cases} c & \text{if } n = 1 \\ U(\lfloor n/2 \rfloor) + U(\lceil n/2 \rceil) + cn & \text{if } n \geqslant 2 \end{cases}$$

  where $c = c_1 + c_2 + c_3$.
- We now show how to solve this recurrence relation using the *recursion tree* method.
- We first assume that n is a power of 2, i.e. $n = 2^h$ for some $h \in \mathbb{N}$.

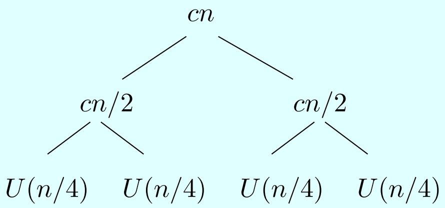
Recursion Tree Method

$$U(n)$$

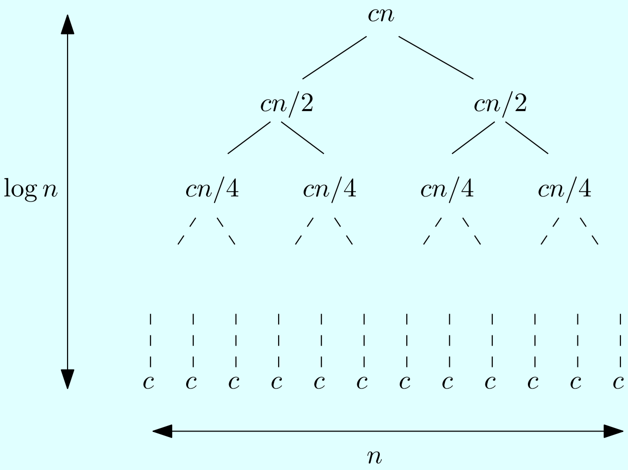
Recursion Tree Method



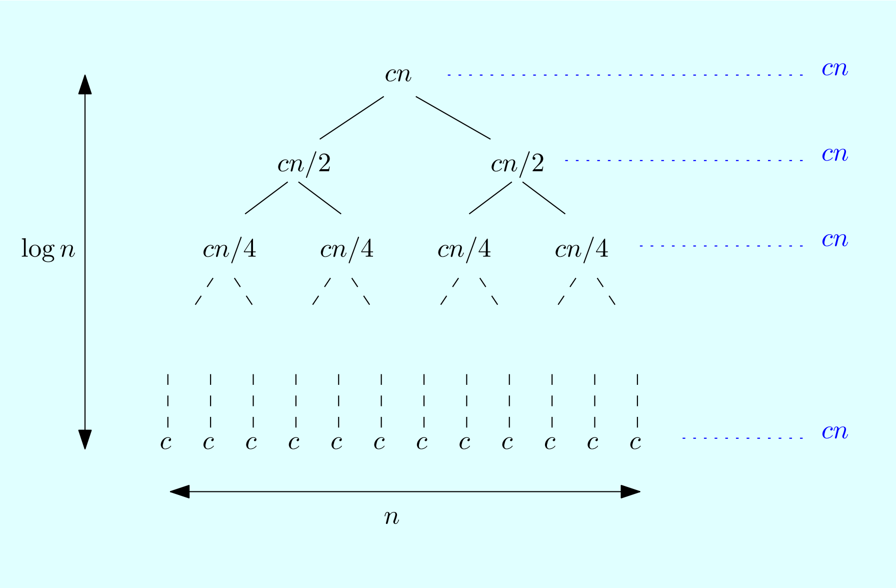
Recursion Tree Method



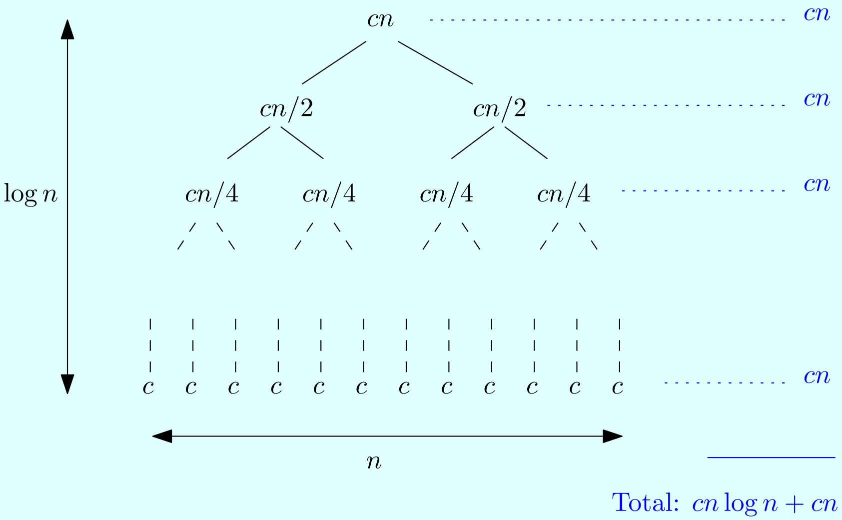
Recursion Tree Method



Recursion Tree Method



Recursion Tree Method



Recursion Tree Method

- Recursion tree method:
  - Expand the recurrence relation into a tree.
  - Add the cost across each level of the tree.
  - Add the costs of all levels.
- Here, the *height* of the tree is $\log n$.
  - In this course, log means $\log_2$
- So there are $1 + \log n$ levels in the tree.
- The cost of each level is cn.
- So $U(n) = cn \log n + cn$.

Technicality

- We only proved $U(n) = cn \log n + cn$ when n is a power of 2.
- Suppose that $2^k < n < 2^{k+1}$.
  - Then the height of the recursion tree is $k+1$. (Proof?)
  - So $U(n) < cn \log n + 2cn$.

Result

- We have just proved that $T(n) < cn \log n + 2cn$ for some constant c.
- The same approach can be used to show that $T(n) > c' n \log n$ for some constant $c' > 0$.
- It can be summarized as follows.

Theorem

MERGE SORT *runs in* $\Theta(n \log n)$ *time.*

- We will argue later this semester that $\Theta(n \log n)$ is the best possible for sorting.

Comparison with INSERTION SORT

- INSERTION SORT runs in $\Theta(n^2)$ time.
  - So for large values of n, MERGE SORT is much faster.
- In the best case, INSERTION SORT runs in $\Theta(n)$ time, while MERGE SORT still runs in $\Theta(n \log n)$ time.
  - So INSERTION SORT is better in the best case.
  - However, when we analyze algorithms, we usually pay more attention to the worst case running time.
- INSERTION SORT is better in terms of memory requirement:
  - The MERGE procedure needs to create two auxiliary arrays of linear size.
  - INSERTION SORT only stores one key outside the input array.
  - We say that INSERTION SORT is *in place*:

Definition

A sorting algorithm is *in place* if it rearranges the numbers within the array A, with at most a constant number of them stored outside the array at any time.