CSE331 Introduction to Algorithms
Lecture 24: Dijkstra's Algorithm

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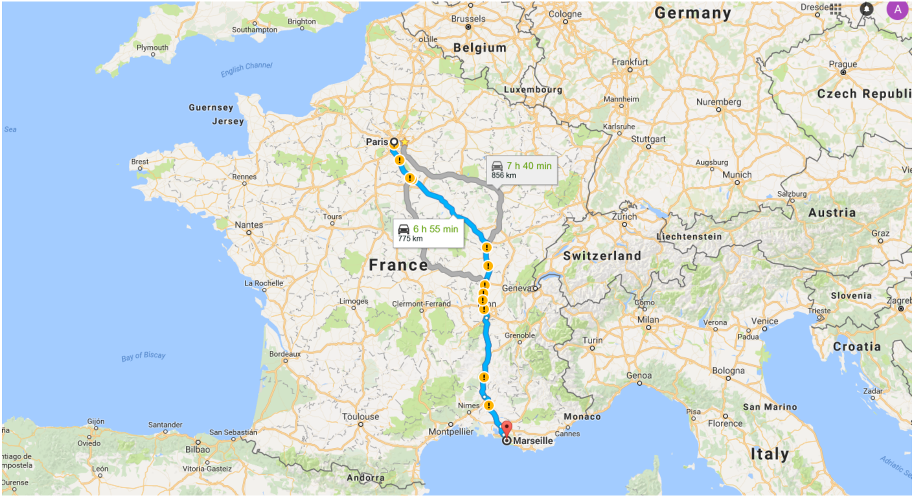
Introduction

- In Lecture 20, we saw that shortest paths in an *unweighted* graph can be computed by BFS in $O(n+m)$ time.
- This lecture presents Dijkstra's algorithm for computing shortest paths in a *weighted* graph.
- (Both algorithms work for directed and undirected graphs.)
- Dijkstra's algorithm can be seen as a greedy algorithm.
- **Reference**: Section 24.3 of the textbook (p 658–)

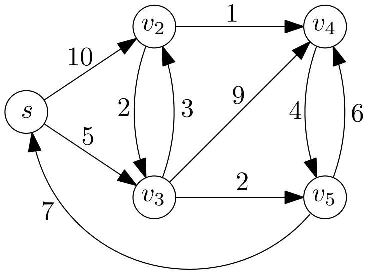
  Introduction to Algorithms by Cormen, Leiserson, Rivest and Stein.

- I will not be following this textbook closely in this lecture.

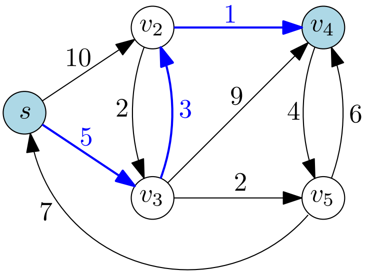
Shortest Paths: Example (Google Maps)



Introduction



Input graph



Shortest path from s to v_4

- The *distance* $d(s, v_4)$ between s and v_4 is the length of a shortest path from s to v_4. So $d(s, v_4) = 9$.

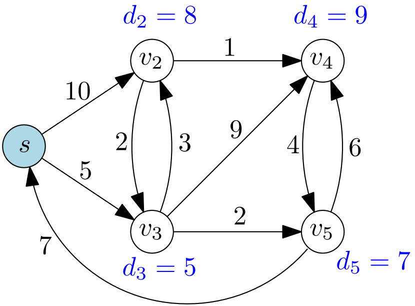
Problem Statement

Problem (Single-source shortest path)

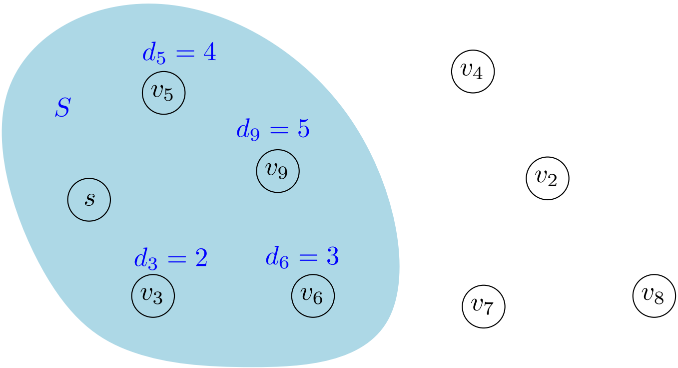
INPUT: a directed graph $G(V, E)$ with vertex set $V = \{v_1, v_2, \dots, v_n\}$, where the *source* node is $s = v_1$. Each edge (v_i, v_j) has a weight $\ell(v_i, v_j) > 0$ called its *length*.

OUTPUT: a shortest path from s to each node v_i, $i = 2, 3, \dots, n$.

- We will show how to compute the distances $d_i = d(s, v_i)$ from s to all the other vertices.

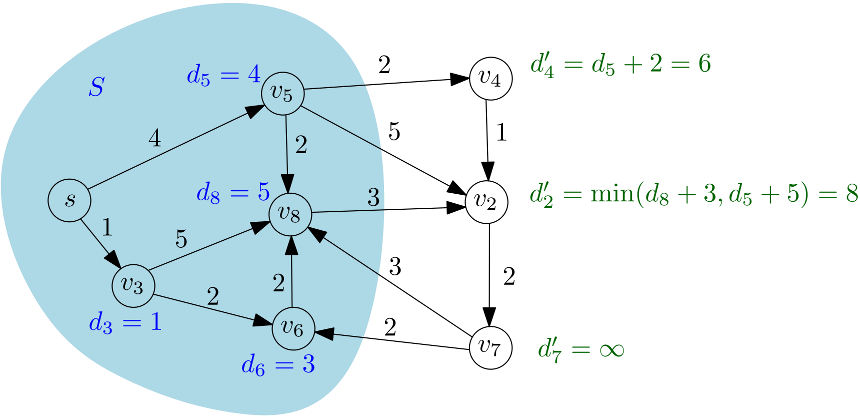


Dijkstra's Algorithm



- During the course of the algorithm, we expand a set S of vertices for which we know the distance from S, i.e. we know d_i for all $v_i \in S$.

Dijkstra's Algorithm



- For each $v_j \notin S$, we maintain the length d'_j of the shortest path from s such that the last edge starts in S. If there is no edge from S to v_j, d'_j is set to ∞.

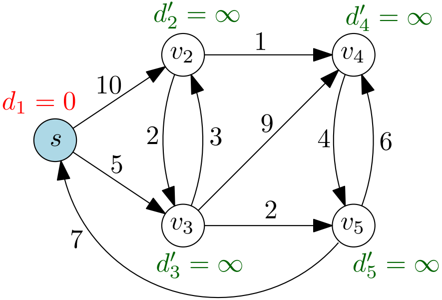
Dijkstra's Algorithm

- The algorithm maintains a set S of *explored* vertices containing s.
- For each $v_i \in S$, we know the length of the shortest path $d_i = d(s, v_i)$ from the source.
- For each $v_j \notin S$ such that there is an edge from a vertex $v_i \in S$ to v_j, we know

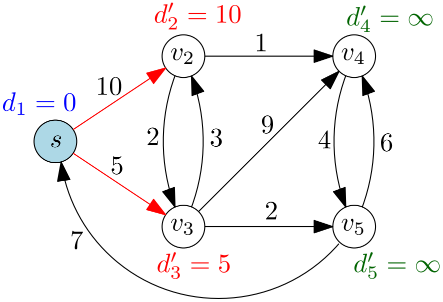
$$d'_j = \min_{v_i \in S:(v_i,v_j) \in E} d_i + \ell(v_i, v_j),$$

  and if there is no such edge, $d'_j = \infty$.
- At each iteration, we insert into S a node $v_i \notin S$ such that d'_i is minimum. We will see that in this case, we have $d_i = d'_i$. We then need to update other values d'_j.

Example

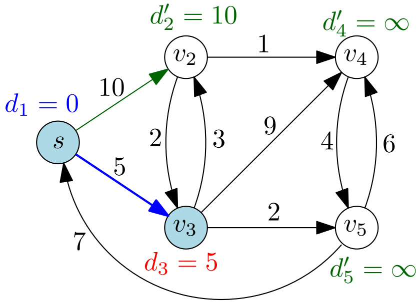


$d_1 = 0$
$d'_2 = \infty$
$d'_4 = \infty$
$d'_3 = \infty$
$d'_5 = \infty$
s
v_2
v_3
v_4
v_5
10
5
7
1
2
3
9
2
4
6



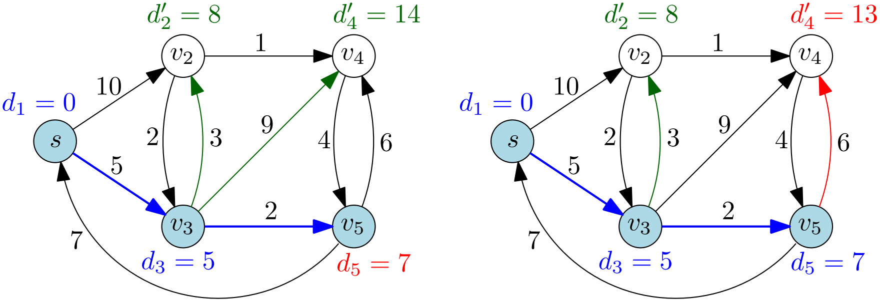
$d_1 = 0$
$d'_2 = 10$
$d'_4 = \infty$
$d'_3 = 5$
$d'_5 = \infty$
s
v_2
v_3
v_4
v_5
10
5
7
1
2
3
9
2
4
6

Example

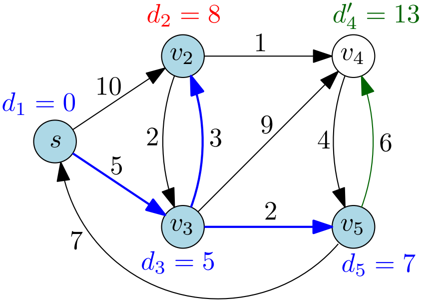


$d'_2 = 10$
$d'_4 = \infty$
$d_1 = 0$
$d_3 = 5$
$d'_5 = \infty$
s
v_2
v_3
v_4
v_5
10
5
2
3
1
9
4
6
2
7
$d'_2 = 8$
$d'_4 = 14$
$d_1 = 0$
$d_3 = 5$
$d'_5 = 7$
s
v_2
v_3
v_4
v_5
10
5
2
3
1
9
4
6
2
7

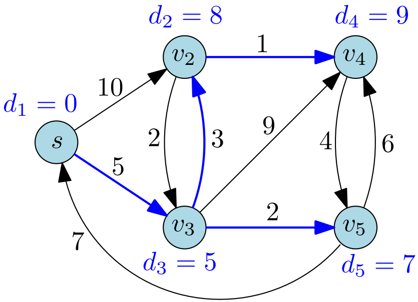
Example


$d_2' = 8$
$d_4' = 14$
$d_1 = 0$
v_2
v_4
s
v_3
v_5
10
1
2
3
9
4
6
5
2
7
$d_3 = 5$
$d_5 = 7$
$d_2' = 8$
$d_4' = 13$
$d_1 = 0$
v_2
v_4
s
v_3
v_5
10
1
2
3
9
4
6
5
2
7
$d_3 = 5$
$d_5 = 7$

Example


$d_2 = 8$
$d_4' = 13$
$d_1 = 0$
$d_3 = 5$
$d_5 = 7$
s
v_2
v_3
v_4
v_5
10
1
2
3
9
4
6
5
2
7
$d_2 = 8$
$d_4' = 9$
$d_1 = 0$
$d_3 = 5$
$d_5 = 7$
s
v_2
v_3
v_4
v_5
10
1
2
3
9
4
6
5
2
7

Example



- The (unique) path from s to v_i in the blue tree is a shortest path.

Dijkstra's Algorithm

Pseudocode

procedure $\textsc{Dijkstra}(G(V,E),\ell)$
    $S \leftarrow \{v_1\}$ ▷ set of explored nodes; $s = v_1$
    $d_1 \leftarrow 0$
    $d'_j \leftarrow \infty$ for all $j \geqslant 2$
    for each edge (s, v_j) **do**
        $d'_j \leftarrow \ell(s, v_j)$
    while $S \neq V$ **do**
        let v_i be the node in $V \setminus S$ such that d'_i is minimum
        $d_i \leftarrow d'_i$
        $S \leftarrow S \cup \{v_i\}$
        for each edge (v_i, v_j) such that $v_j \notin S$ **do**
            $d'_j \leftarrow \min(d'_j, d_i + \ell(v_i, v_j))$

Proof of Correctness

- We prove that the algorithm is correct using the loop invariants below. Correctness of the algorithm then follows immediately from Invariant (a).

Lemma

At the beginning of each iteration of the **while** *loop, the following hold:*

(a) *For every $v_p \in S$, the value d_p is the length $d(s, v_p)$ of a shortest path from s to v_p.*

(b) *If there is an edge from a node of S to a node $v_q \notin S$, then d'_q is the length of a shortest path from s to v_q, under the constraint that the last edge starts from S.*

Proof of the Lemma

- Lines 2–6 ensure that the invariants are true at the beginning of the first iteration.
- Now suppose it is true at the beginning of the current iteration.
- Let v_i be the node chosen at Line 8.
- We first prove that $d'_i = d(s, v_i)$.
- Let P be a path from s to v_i.
- Let v_b be the first node in P that is not in S, and let v_a be the vertex just before v_b along P.
- Then the length $\ell(P)$ of P satisfies:

$$\begin{aligned}
\ell(P) &\geqslant d_a + \ell(v_a, v_b) && \text{because } d_a \text{ is the length of a} \\
&&& \text{shortest path from } s \text{ to } v_a \\
&\geqslant d'_b && \text{by (b)} \\
&\geqslant d'_i && \text{because } v_i \text{ minimizes } d'_i
\end{aligned}$$

Proof of the Lemma

- We have shown that any path from s to v_i has length at least d'_i.
- On the other hand, Invariant (b) ensures that one such path has length d'_i.
- It means that $d'_i = d(s, v_i)$, and thus Invariant (a) will be true at the end of this iteration.
- Then Lines 11–12 ensure that Invariant (b) is maintained, by handling the outgoing edges from the vertex v_i that was inserted into S.

Efficient Implementation and Analysis

- The while loop is iterated n times, when $n = |V|$.
- So Line 8 is executed n times.
- We store the values d'_j in a heap-based priority queue $\mathcal{Q}$, which allows to execute Line 8 in $O(\log n)$ time.
- Similarly, Line 12 takes $O(\log n)$ time using a $\textsc{ChangeKey}$ operation.
- Line 12 is executed at most once for each edge (v_i, v_j), as v_i is inserted into S at most once, so in total Line 12 is executed $m = |E|$ times.

Theorem

Dijkstra's algorithm can be implemented to run in time $O((n+m)\log n)$ time, where n is the number of vertices and m is the number of edges.

Concluding Remarks

- Dijkstra's algorithm can be seen as a greedy algorithm as at each step, we pick the closest unexplored vertex.
- We only saw how to compute d_i. A shortest path from s to v_i can be found by simple modifications of the algorithm, as we did for our dynamic programming algorithms. Same as BFS, we can construct a spanning tree (blue in the figures) such that the unique path $s \rightsquigarrow v_i$ in this tree is the shortest path within the input graph.
- Dijkstra's algorithm works when all edge weights are positive, but it may fail for negative weights. (See exercise set.)