CSE331: Introduction to Algorithms
Notes on Lecture 4: Merge Sort

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It is mentioned in the slides that the height of the recursion tree for MERGE SORT is $k+1$ whenever $2^k < n \leqslant 2^{k+1}$. Here is a proof.

We prove it by induction on k. Let $h(n)$ denote the height of the recursion tree on an input array of size n. We want to prove the following for all $k \geqslant 1$:

$$P(k): \; h(n) = k+1 \text{ for all } n \text{ such that } 2^k < n \leqslant 2^{k+1} \tag{1}$$

Base case: When $k=0$, then $1 < n \leqslant 2$ so $n=2$. In this case, there is exactly one level of recursion to two subarrays of size 1, and thus $h(2)=1$.

Inductive step: Suppose that $P(q)$ is true for all $q<k$, and we want to prove that $P(k)$ is also true. Suppose $2^k < n \leqslant 2^{k+1}$. Then MERGE SORT recurses on two subarrays of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. So we have

$$h(n) = 1 + \max\Big(h(\lfloor n/2 \rfloor), h(\lceil n/2 \rceil)\Big) \tag{2}$$

Since $2^k < n \leqslant 2^{k+1}$, we have $2^{k-1} < \lceil n/2 \rceil \leqslant 2^k$. As $P(k-1)$ is true by our inductive hypothesis, it means that $h(\lceil n/2 \rceil) = k$. We also have $2^{k-1} \leqslant \lfloor n/2 \rfloor \leqslant 2^k$, and since $P(k-2)$ and $P(k-1)$ are true, it implies that $h(\lfloor n/2 \rfloor) = k$ or $k-1$. So Equation (1) implies that $h(n) = 1+k$.