CSE331: Introduction to Algorithms
Notes on Lecture 15: Can we sort in linear time?

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October 28, 2020

Proof that $\log(n!) = \Theta(n \log n)$

One way of proving it is to use Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

A more elementary solution is the following. We first prove that $\log(n!) = O(n \log n)$.

$$\begin{aligned} n! &= 1 \cdot 2 \ldots (n-1) \cdot n \\ &\leqslant n^n \end{aligned}$$

and thus

$$\begin{aligned} \log(n!) &\leqslant \log(n^n) \\ &= n \log n. \end{aligned}$$

To prove the lower bound $\log(n!) = \Omega(n \log n)$, let us write $m = \lfloor n/2 \rfloor$. Then we have

$$n! = 1 \cdot 2 \ldots m \cdot \underbrace{(m+1) \ldots n}_{n-m \text{ numbers}}.$$

Since $n - m \geqslant m$, it follows that $n! > m^m$ and thus $\log(n!) \geqslant m \log m$. As $m \geqslant \frac{n}{2} - \frac{1}{2}$, it implies that $\log(n!) = \Omega(n \log n)$.