Advanced Algorithms
Lecture 6: Maximum Flow I

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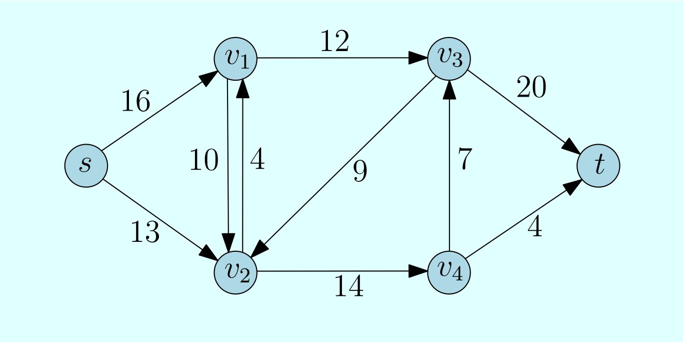
March 18, 2021

Reference

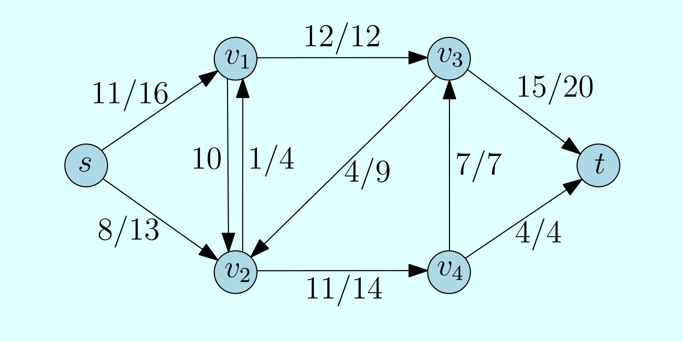
- Reference: Chapter 26 in Introduction to Algorithms by Cormen, Leiserson, Rivest and Stein.
  - These slides are based on the *2nd edition* (2001), also available at the library.
  - The 3rd edition uses a different convention: If edge $(u, v) \in E$ then $(v, u) \notin E$, and then flow conservation is written differently and skew symmetry is irrelevant.

Flow Networks



- A *flow network* $G = (V, E)$ is a directed graph.
- Each edge (u, v) is weighted by a non-negative *capacity* $c(u, v) \geqslant 0$.
  - If $(u, v) \notin E$, then $c(u, v) = 0$.
- Two special vertices: the *source* s and the *sink* t.
- For each $v \in V$, there is a path $s \rightsquigarrow v \rightsquigarrow t$.

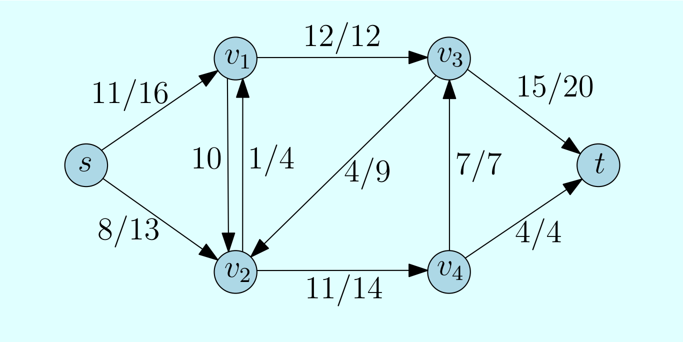
Flows



A *flow* in G is a function $f : V \times V \to \mathbb{R}$ such that:

- $\forall u, v \in V,\ f(u,v) \leqslant c(u,v)$. (*Capacity constraint*)
- $\forall u, v \in V,\ f(u,v) = -f(v,u)$. (*Skew symmetry*)
- $\forall u \in V \setminus \{s,t\},\ \sum_{v \in V} f(u,v) = 0$. (*Flow conservation*)

Terminology

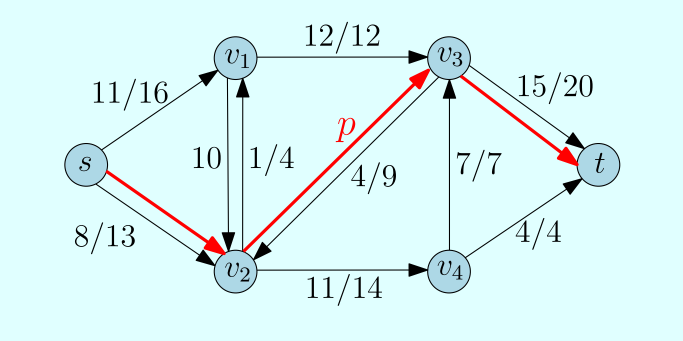


- $f(u, v)$ is called the *flow* from u to v.
- The *value* of a flow is $|f| = \sum_{v \in V} f(s, v)$.
- The *maximum-flow problem* is to find a flow of maximum value in a flow network.

Augmenting Path

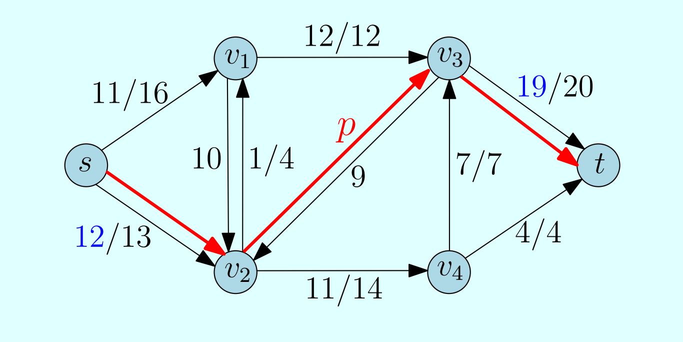
Definition

A *simple path* in a graph is a path with no repeated vertices.



An *augmenting path* is a simple path $p : s \rightsquigarrow t$ along which we can send more flow.

Augmenting Path



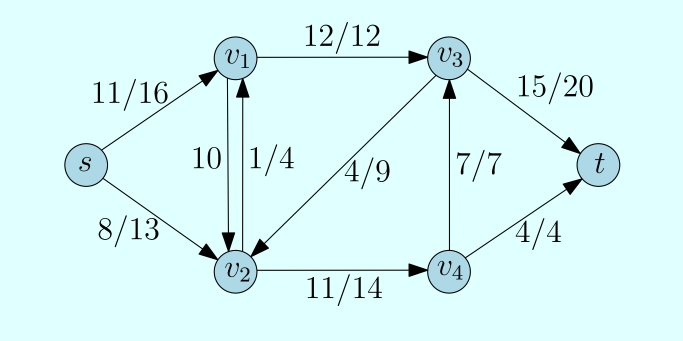
Result after sending 4 units of flow along the augmenting path p.

The Ford-Fulkerson Method

Ford-Fulkerson method for maximum flow

1: initialize flow f to 0
2: **while** there exists an augmenting path p **do**
3: augment flow f along p.
4: **return** f

Residual Network

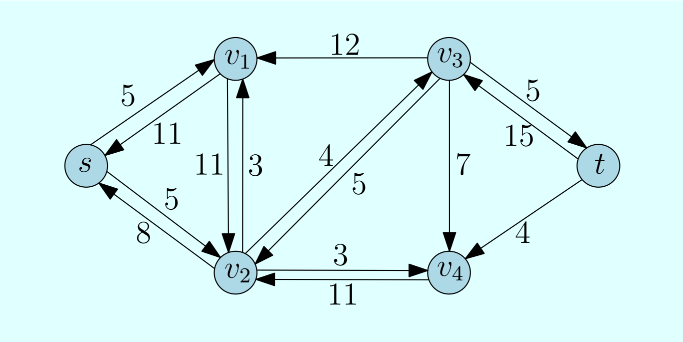


A flow network G and a flow f.

The *residual capacity* of (u, v) is $c_f(u, v) = c(u, v) - f(u, v)$.

- Here, $c_f(s, v_2) = 5$ and $c_f(v_2, v_3) = 0 - (-4) = 4$.
- Intuitively, the residual capacity $c_f(u, v)$ is the additional amount of flow we can push from u to v.

Residual Network

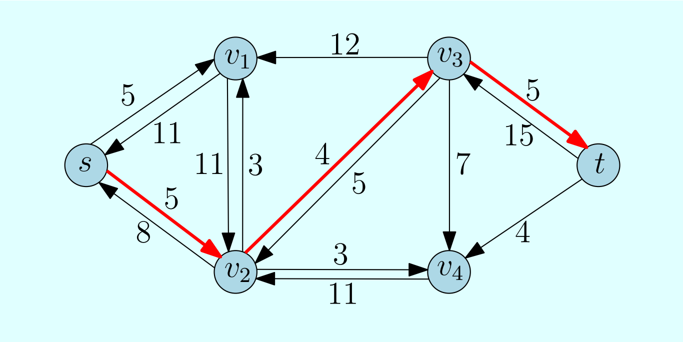


The *residual network* $G_f(V, E_f)$, with edge set

$$E_f = \{(u, v) \in V \times V \mid c_f(u, v) > 0\}.$$

Residual Network

The *residual capacity of a path* p is $c_f(p) = \min\{c_f(u, v) \mid (u, v) \text{ is on } p\}$.

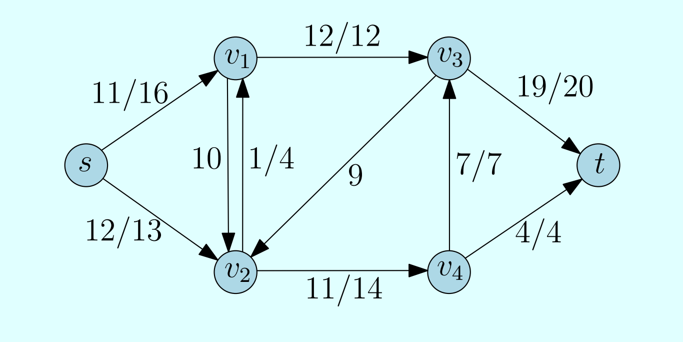


The augmenting path p, with residual capacity 4.

Property

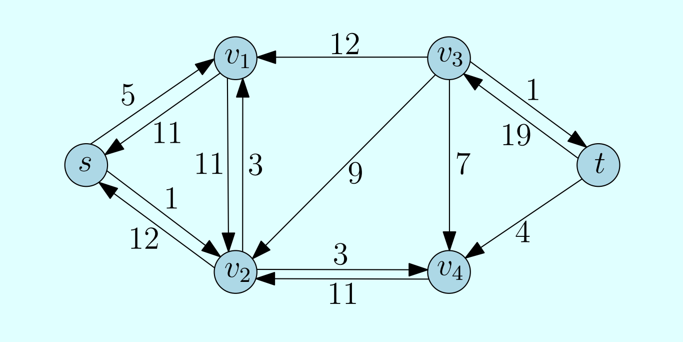
An augmenting path in G is a simple path $p : s \rightsquigarrow t$ such that $c_f(p) > 0$, or equivalently, it is a simple path $p : s \rightsquigarrow t$ in G_f.

Residual Network



The flow after augmenting p by its residual capacity 4.

Residual Network



The residual network after augmenting p by its residual capacity 4.

There is no augmenting path now, the Ford-Fulkerson method returns this flow.

Flow Sums

Definition

Let f_1 and f_2 be flows in G. Let $f_1 + f_2 : V \times V \to \mathbb{R}$ be the function such that $(f_1 + f_2)(u, v) = f_1(u, v) + f_2(u, v)$ for all $u, v \in V$. If $f_1 + f_2$ is a flow in G, then we say that $f_1 + f_2$ is the *flow sum* of f_1 and f_2.

- Which flow property can fail for $f_1 + f_2$?

Lemma

Let f be a flow in the flow network G. Let f' be a flow in the residual network G_f. Then $f + f'$ is a flow in G with value $|f + f'| = |f| + |f'|$.

Proof done in class.

Augmenting Paths

Let $G = (V, E)$ be a flow network. Let f be a flow in G, and let p be an augmenting path in G_f. Define $f_p : V \times V \to \mathbb{R}$ by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \text{ is on } p, \\ -c_f(p) & \text{if } (v, u) \text{ is on } p, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma

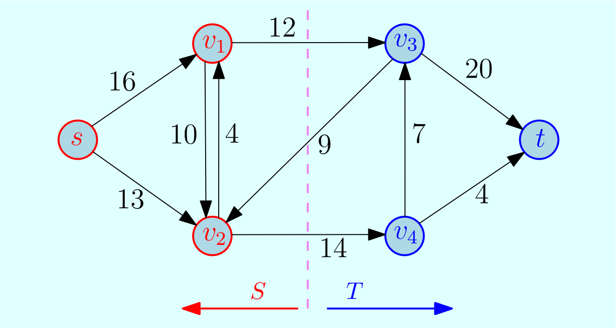
The function f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

Proof done in class.

Corollary

Let $f' : V \times V \to \mathbb{R}$ be defined by $f' = f + f_p$. Then f' is a flow in G with value $|f'| = |f| + |f_p| > |f|$.

Cuts of Flow Networks

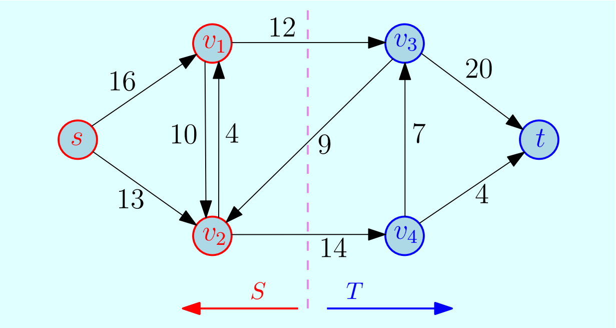


Definition

A *cut* (S, T) of a flow network $G = (V, E)$ is a partition of V into S and $T = V \setminus S$ such that $s \in S$ and $t \in T$.

Here $(S, T) = (\{s, v_1, v_2\}, \{v_3, v_4, t\})$.

Cuts of Flow Networks

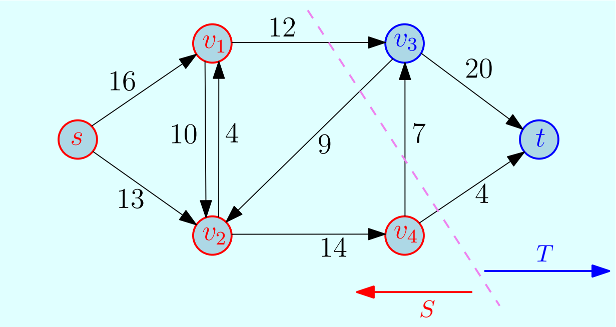


Definition

The *capacity* of a cut (S, T) is $c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)$.

Here $c(S, T) = 12 + 14 = 26$.

Cuts of Flow Networks

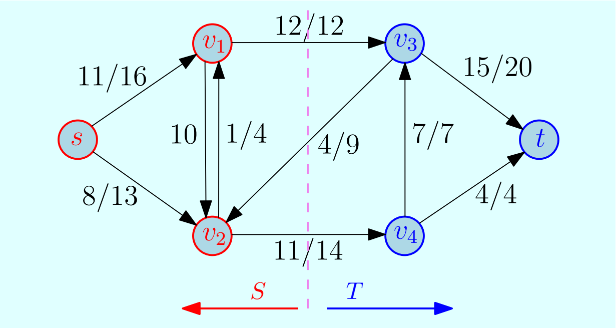


Definition

A *minimum cut* is a cut (S, T) with minimum capacity.

Here the minimum cut (S, T) has capacity $c(S, T) = 12 + 7 + 4 = 23$.

Cuts of Flow Networks



Definition

The *net flow* across a cut (S, T) is $f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v)$.

Here $f(S, T) = 12 + 11 - 4 = 19$.

Cuts of Flow Networks

Lemma

For any cut (S, T), the net flow $f(S, T)$ across (S, T) is equal to the value $|f|$ of the flow.

Proof (sketch).

For any $X, Y \subset V$, we denote $f(X, Y) = \sum_{u \in X} \sum_{v \in Y} f(u, v)$.

$$\begin{aligned} f(S, T) &= f(S, V) - f(S, S) \\ &= f(S, V) \\ &= f(\{s\}, V) + f(S \setminus \{s\}, V) \\ &= f(\{s\}, V) \\ &= |f| \end{aligned}$$

Cuts of Flow Networks

Corollary (1)

The flow $\sum_{u \in V} f(u, t)$ *into the sink is equal to* $|f|$.

Corollary (2)

The value $|f|$ *of any flow* f *is at most the capacity* $c(S, T)$ *of any cut* (S, T).