Advanced Algorithms
Lecture 9
Linear Programming: Standard and Slack Forms

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Introduction

- Assignment 1 is due on Friday.
- Reference: Chapter 29.1 in Introduction to Algorithms by Cormen, Leiserson, Rivest and Stein.

Standard Form

- A LP in *standard form* is as follows:

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, & i = 1, \dots, m \\
& x_j \geqslant 0, & j = 1, \dots, n
\end{array}
$$

- The constraints $x_j \geqslant 0$ are the *nonnegativity contraints*.
- Using the matrix $A = (a_{ij})$ and the vectors $b = (b_i)$, $c = (c_j)$, and $x = (x_j)$, the standard form can be written:

$$
\begin{array}{ll}
\text{maximize} & c^T x \\
\text{subject to} & Ax \leqslant b \\
& x \geqslant 0.
\end{array}
$$

Standard Form

Theorem

Any linear program can be written in standard form.

Proof.

- Minimizing $c^T x$ is equivalent to maximizing $(-c)^T x$.
- We can always reduce to the case where the variables satisfy the nonnegativity constraints (done in class).
- The constraint $\sum_j a_{ij} x_j = b_i$ is equivalent to $\sum_j a_{ij} x_j \leqslant b_i$ and $\sum_j a_{ij} x_j \geqslant b_i$.
- The constraint $\sum_j a_{ij} x_j \geqslant b_i$ is equivalent to $\sum_j (-a_{ij}) x_j \leqslant -b_i$.

$\square$

Standard Form

Example

Convert the LP below into standard form.

$$
\begin{array}{rl}
\text{minimize} & -2x_1 + 3x_2 \\
\text{subject to} & x_1 + x_2 = 7 \\
& x_1 - 2x_2 \leqslant 4 \\
& x_1 \geqslant 0
\end{array}
$$

Answer

$$
\begin{array}{rl}
\text{maximize} & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} & x_1 + x_2 - x_3 \leqslant 7 \\
& -x_1 - x_2 + x_3 \leqslant -7 \\
& x_1 - 2x_2 + 2x_3 \leqslant 4 \\
& x_1, x_2, x_3 \geqslant 0
\end{array}
$$

(Variable x_2 in the original LP has been replaced with $x_2 - x_3$.)

Slack Form

- A LP can be rewritten with only equality constraints.
- For instance, the LP in standard form from the previous slide can be rewritten as:

$$
\begin{array}{rrcl}
\text{maximize} & & & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} & x_4 & = & 7 - x_1 - x_2 + x_3 \\
& x_5 & = & -7 - x_1 + x_2 - x_3 \\
& x_6 & = & 4 - x_1 + 2x_2 - 2x_3 \\
& & & x_1, x_2, x_3, x_4, x_5, x_6 \geqslant 0
\end{array}
$$

- We rewrite it in a more concise way, called *slack form*:

$$
\begin{array}{rcl}
z & = & 2x_1 - 3x_2 + 3x_3 \\
x_4 & = & 7 - x_1 - x_2 + x_3 \\
x_5 & = & -7 - x_1 + x_2 - x_3 \\
x_6 & = & 4 - x_1 + 2x_2 - 2x_3
\end{array}
$$

Converting a LP in Standard Form into Slack Form

- We start with a LP in standard form:

$$\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \qquad && i = 1, \dots, m \\
& x_j \geqslant 0, && j = 1, \dots, n
\end{aligned}$$

- For $i = 1, \dots, m$, we replace the ith constraint with

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j.$$

where x_{n+i} is a new variable, called a *slack variable*.

Converting a LP in Standard Form into Slack Form

- We also add the nonnegativity constraints

$$x_{n+i} \geqslant 0, \quad i = 1, \dots, m.$$

- The objective function is represented by a new variable

$$z = \sum_{j=1}^{n} c_j x_j.$$

- So the slack form of the LP in previous slide is:

$$z = \sum_{j=1}^{n} c_j x_j$$

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \dots m.$$

Slack Form

- The variables on the left-hand side are called *basic variables*.
  - We denote by B the set of their indices.
  - So in previous slide's program, $B = \{n+1, \ldots, n+m\}$.
- The variables on the right-hand side are called *nonbasic variables*.
  - We denote by N the set of their indices.
  - So in previous slide's program, $N = \{1, \ldots, n\}$.

Slack Form

- We will need a more general slack form for next lecture.
- N and B will not necessarily be $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$.
- So we only require:
  - $N \cup B = \{1, \ldots, n+m\}$,
  - $|N| = n$ and $|B| = m$,
  - and thus $N \cap B = \emptyset$.
- We also add a constant term ν to the objective function.
- So the general form of a linear program in slack form is:

$$z = \nu + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j, \qquad i \in B.$$

- This slack form is defined by the tuple (N, B, A, b, c, ν).

Slack Form

- Example:

$$\begin{aligned} z &= 28 - \tfrac{1}{6}x_3 - \tfrac{1}{6}x_5 - \tfrac{2}{3}x_6 \\ x_1 &= 8 + \tfrac{1}{6}x_3 + \tfrac{1}{6}x_5 - \tfrac{1}{3}x_6 \\ x_2 &= 4 - \tfrac{8}{3}x_3 - \tfrac{2}{3}x_5 + \tfrac{1}{3}x_6 \\ x_4 &= 18 - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_5 \end{aligned}$$

- Here $\nu = 28$, $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$,

$$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{8}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \qquad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} \\ -\frac{1}{6} \\ -\frac{2}{3} \end{pmatrix}.$$