Advanced Algorithms
Lecture 10
The Simplex Algorithm I

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April 1, 2021

Introduction

- Reminder: Assignment 2 is due on Friday.
- The *simplex algorithm* is a practical algorithm for linear programming.
- It does not run in polynomial time: examples are known where it runs in exponential time. But in practice, it usually runs in polynomial time.
- Polynomial time algorithms are known, but they are only *weakly* polynomial, i.e. their running time depends on the precision of the input.
- Finding a strongly polynomial algorithm for LP is an important open problem.
- Reference: Chapter 29.3 of the textbook Introduction to Algorithms by Cormen, Leiserson, Rivest and Stein.

Example

We want to solve the following linear program, given in standard form:

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & x_1 & + & x_2 & + & 3x_3 & \leqslant & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leqslant & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leqslant & 36 \\
 & & & & & x_1, x_2, x_3 & \geqslant & 0
\end{array}
$$

We first convert it into slack form:

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Example

- The slack form above has basic variables x_4, x_5, x_6 and nonbasic variables x_1, x_2, x_3.
- So $N = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$.
- The *basic solution* corresponding to this slack form is

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 30, 24, 36)$$

  with value $z = 0$.
  - In the basic solution, all nonbasic variables are set to 0.
- Approach: Modify N and B, keeping an equivalent program, and increasing the value of the basic solution.

Example

$$\begin{array}{rcrcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{array}$$

- In our slack form, increasing x_1 would increase z.
- Due to the 3rd constraint, we can increase x_1 to 9 at most.
- We move x_1 to LHS in 3rd constraint, and obtain:

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6.$$

- We now replace x_1 with $9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$ in the other equations and obtain:

Example

$$
\begin{array}{rcrcrcrcr}
z & = & 27 & + & \frac{1}{4}x_2 & + & \frac{1}{2}x_3 & - & \frac{3}{4}x_6 \\
x_1 & = & 9 & - & \frac{1}{4}x_2 & - & \frac{1}{2}x_3 & - & \frac{1}{4}x_6 \\
x_4 & = & 21 & - & \frac{3}{4}x_2 & - & \frac{5}{2}x_3 & + & \frac{1}{4}x_6 \\
x_5 & = & 6 & - & \frac{3}{2}x_2 & - & 4x_3 & + & \frac{1}{2}x_6
\end{array}
$$

- This operation is called a *pivot*.
- Now $N = \{2, 3, 6\}$ and $B = \{1, 4, 5\}$.
    - x_1 has become a basic variable, and x_6 has become nonbasic.
    - x_1 is called the *entering variable*, and x_6 is the *leaving variable*.
- The basic solution is $(x_1, x_2, x_3, x_4, x_5, x_6) = (9, 0, 0, 21, 6, 0)$.
- The value of the objective function at this solution is 27.
    - We can read it from the top line of this new slack form.
    - Or we can substitute $x_1 = 9$, $x_2 = 0$ and $x_3 = 0$ in Slide 4.
- It will always be the case: At each step of the simplex algorithm, the new LP is equivalent to the LP at the previous step.

Example

- We can now increase x_2 or x_3, but not x_6 as its coefficient is negative and thus it would decrease the value of the objective function.
- We choose x_3.
- The limiting constraint is the last one, with $x_3 = \frac{3}{2}$.
- So we have

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6.$$

and

$$\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{1}{16}x_2 & - & \frac{1}{8}x_5 & - & \frac{11}{16}x_6 \\
x_1 & = & \frac{33}{4} & - & \frac{1}{16}x_2 & + & \frac{1}{8}x_5 & - & \frac{5}{16}x_6 \\
x_3 & = & \frac{3}{2} & - & \frac{3}{8}x_2 & - & \frac{1}{4}x_5 & + & \frac{1}{8}x_6 \\
x_4 & = & \frac{69}{4} & + & \frac{3}{16}x_2 & + & \frac{5}{8}x_5 & - & \frac{1}{16}x_6
\end{array}$$

Example

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{1}{16}x_2 & - & \frac{1}{8}x_5 & - & \frac{11}{16}x_6 \\
x_1 & = & \frac{33}{4} & - & \frac{1}{16}x_2 & + & \frac{1}{8}x_5 & - & \frac{5}{16}x_6 \\
x_3 & = & \frac{3}{2} & - & \frac{3}{8}x_2 & - & \frac{1}{4}x_5 & + & \frac{1}{8}x_6 \\
x_4 & = & \frac{69}{4} & + & \frac{3}{16}x_2 & + & \frac{5}{8}x_5 & - & \frac{1}{16}x_6
\end{array}
$$

- Last step: Increase x_2 by 4:

$$
\begin{array}{rcrcrcrcr}
z & = & 28 & - & \frac{1}{6}x_3 & - & \frac{1}{6}x_5 & - & \frac{2}{3}x_6 \\
x_1 & = & 8 & + & \frac{1}{6}x_3 & + & \frac{1}{6}x_5 & - & \frac{1}{3}x_6 \\
x_2 & = & 4 & - & \frac{8}{3}x_3 & - & \frac{2}{3}x_5 & + & \frac{1}{3}x_6 \\
x_4 & = & 18 & - & \frac{1}{2}x_3 & + & \frac{1}{2}x_5 & &
\end{array}
$$

- The optimal solution is $(x_1, x_2, x_3, x_4, x_5, x_6) = (8, 4, 0, 18, 0, 0)$ with value 28. Proof?
- The solution to the original problem is $(x_1, x_2, x_3) = (8, 4, 0)$ and the optimal value is also 28.

General Case

- LP in slack form:

$$z = \nu + \sum_{j \in N} c_j x_j$$
$$x_i = b_i - \sum_{j \in N} a_{ij} x_j, \qquad i \in B.$$

- Next slide: Pseudocode for the Pivot operation, the input is $(N, B, A, b, c, \nu, \ell, e)$ where e is the index of the entering variable, and ℓ the leaving variable.
- The output is $(\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{\nu})$.

Pivot$(N, B, A, b, c, \nu, \ell, e)$

$\hat{b}_e \leftarrow b_\ell / a_{\ell e}$ ▷ Coefs. of entering variable x_e

for each $j \in N \setminus \{e\}$ **do**

  $\hat{a}_{ej} \leftarrow a_{\ell j} / a_{\ell e}$

$\hat{a}_{e\ell} \leftarrow 1/a_{\ell e}$

for each $i \in B \setminus \{\ell\}$ **do** ▷ Coefs. of other constraints

  $\hat{b}_i \leftarrow b_i - a_{ie}\hat{b}_e$

  for each $j \in N \setminus \{e\}$ **do**

    $\hat{a}_{ij} \leftarrow a_{ij} - a_{ie}\hat{a}_{ej}$

  $\hat{a}_{i\ell} \leftarrow -a_{ie}\hat{a}_{e\ell}$

$\hat{\nu} \leftarrow \nu + c_e\hat{b}_e$ ▷ Coefs. of objective function

for each $j \in N \setminus \{e\}$ **do**

  $\hat{c}_j \leftarrow c_j - c_e\hat{a}_{ej}$

$\hat{c}_\ell \leftarrow -c_e\hat{a}_{e\ell}$

$\hat{N} \leftarrow N \setminus \{e\} \cup \{\ell\}$, $\hat{B} \leftarrow B \setminus \{\ell\} \cup \{e\}$

return $(\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{\nu})$

General Case

- We assume that we start with a slack form (N, B, A, b, c, ν) whose basic solution is feasible.
  - We explain at the end of this lecture how to find this solution.
- If all coefficients are negative in the objective function, we just return the basic solution restricted to $(x_1, \ldots, x_n)$.
- Otherwise we increase as much as possible one of the variables x_e with nonnegative coefficient.
  - If we can increase to ∞, then return *unbounded*.
  - Otherwise perform a pivot using x_e as entering variable, and using the leaving variable x_ℓ corresponding to the saturated constraint.
- *Next slide: pseudocode.*

General Case

Simplex(A, b, c)

$(N, B, A, b, c, \nu) \leftarrow$ Initialize-Simplex(A, b, c)
while $\exists j : c_j > 0$ **do**
  Choose e such that $c_e > 0$
  for each $i \in B$ **do**
    if $a_{ie} > 0$ **then** $\Delta_i \leftarrow b_i / a_{ie}$
    else $\Delta_i \leftarrow \infty$
  Choose ℓ that minimizes Δ_ℓ
  if $\Delta_\ell = \infty$ **then** **return** unbounded
  else $(N, B, A, b, c, \nu) \leftarrow$ Pivot($N, B, A, b, c, \nu, \ell, e$)
for $i \leftarrow 1, n$ **do**
  if $i \in B$ **then** $\bar{x}_i \leftarrow b_i$
  else $\bar{x}_i \leftarrow 0$
return $(\bar{x}_1, \ldots, \bar{x}_n)$

Degeneracy

- By construction, the value of the objective function never decreases during the course of the simplex algorithm.
- But in some cases, it may remain the same after one step:

$$\begin{array}{rcccccccc} z & = & & & x_1 & + & x_2 & + & x_3 \\ x_4 & = & 8 & - & x_1 & - & x_2 & & \\ x_5 & = & & & & & x_2 & - & x_3 \end{array}$$

- Assume we choose $e = 1$, and thus $\ell = 4$. We get:

$$\begin{array}{rcccccccc} z & = & 8 & & & + & x_3 & - & x_4 \\ x_1 & = & 8 & - & x_2 & & & - & x_4 \\ x_5 & = & & & x_2 & - & x_3 & & \end{array}$$

- The value of the basic solution is 8.

Degeneracy

$$\begin{array}{rcrcrcrcr} z & = & 8 & & & + & x_3 & - & x_4 \\ x_1 & = & 8 & - & x_2 & & & - & x_4 \\ x_5 & = & & & x_2 & - & x_3 & & \end{array}$$

- At this point, we can only choose $e = 3$ and $\ell = 5$:

$$\begin{array}{rcrcrcrcr} z & = & 8 & + & x_2 & - & x_4 & - & x_5 \\ x_1 & = & 8 & - & x_2 & - & x_4 & & \\ x_3 & = & & & x_2 & & & - & x_5 \end{array}$$

- The value of the basic solution is still 8.
- So we may not make progress at each step of the simplex algorithm. This is called *degeneracy*.
- Here, fortunately, if we pivot again, we have $e = 2$ and $\ell = 1$, and then the value increases to 16.

Cycling

- In some degenerate cases however, the simplex algorithm may go back to the same slack form repeatedly.
- Then the value does not increase, and the algorithm does not terminate.
- This is called *cycling*.
- An example with 6 variables and 3 equations is known.
- Cycling can be avoided by a careful choice of the pivot.
- For instance, using *Bland's rule*: Choose the entering variable with smallest index, and then the leaving variable with smallest index.

Theorem

Using Bland's rule, the simplex algorithm never cycles.

- We will not prove it in this course.