CSE515 Advanced Algorithms
Lecture 12: Interior Point Methods

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Introduction

- Previous lecture: the *simplex algorithm* is a practical algorithm for linear programming, but it does not run in (worst-case) polynomial time.
- Walks along edges of the feasible region, moving from vertex to vertex.

- In this lecture, I present a polynomial-time algorithm.
- It is an *interior-point* method.
- As opposed to the simplex algorithm, it walks in the *interior* of the feasible region.
- I will not give proofs. The goal is to show you how this algorithm works.
- Reference: Convex Optimization by Boyd and Vandenberghe. (Freely available online.)

Newton's Method

- Newton's method is a very efficient *unconstrained* minimization method for smooth convex functions.
- (Different from Newton's method for finding roots.)
- It finds the minimum over $\mathbb{R}^n$ of a twice-differentiable convex function $f : \mathbb{R}^n \to \mathbb{R}$. We denote

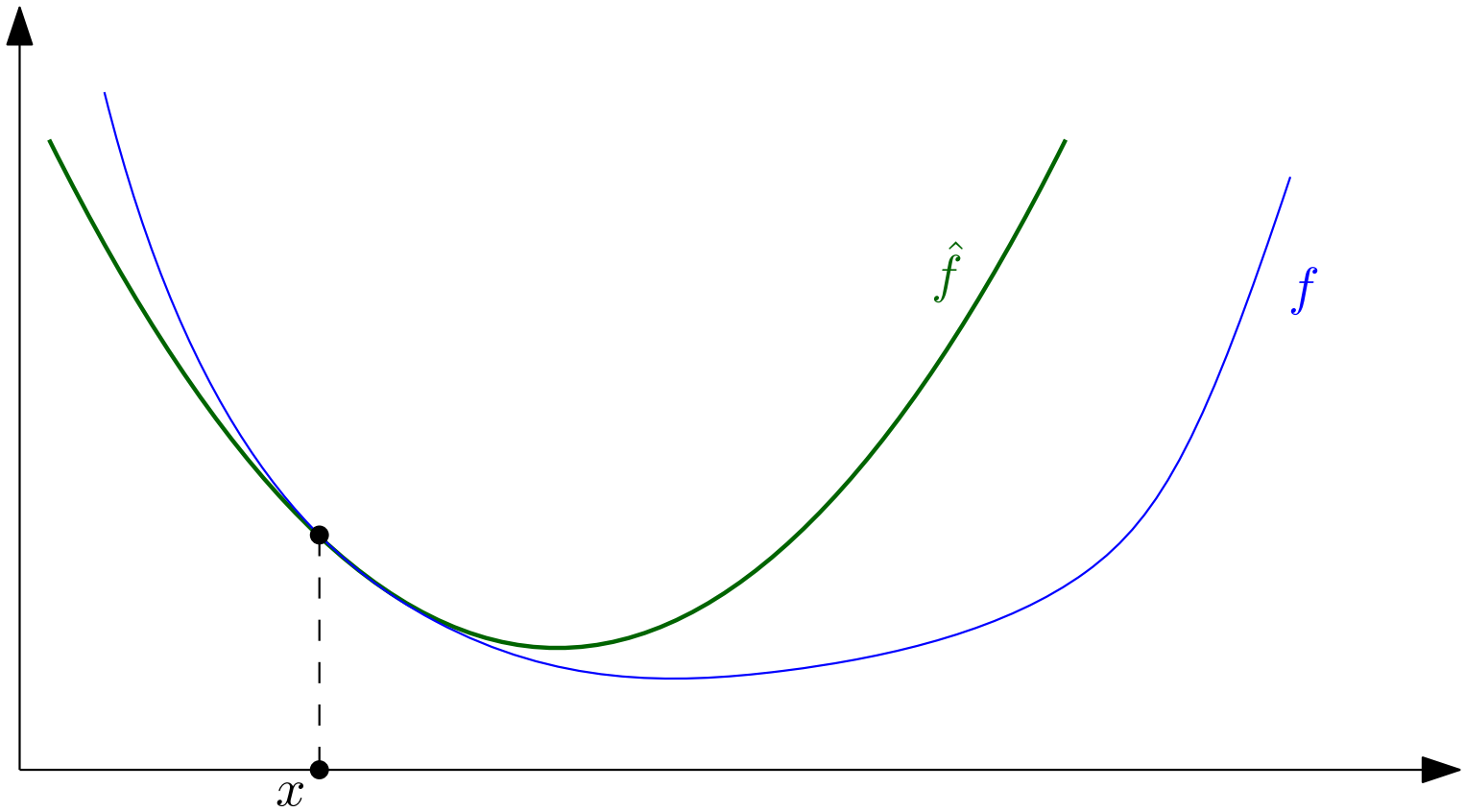
$$p^* = \min_{x \in \mathbb{R}^n} (f(x))$$

and

$$x^* = \arg\min_{x \in \mathbb{R}^n} (f(x)),$$

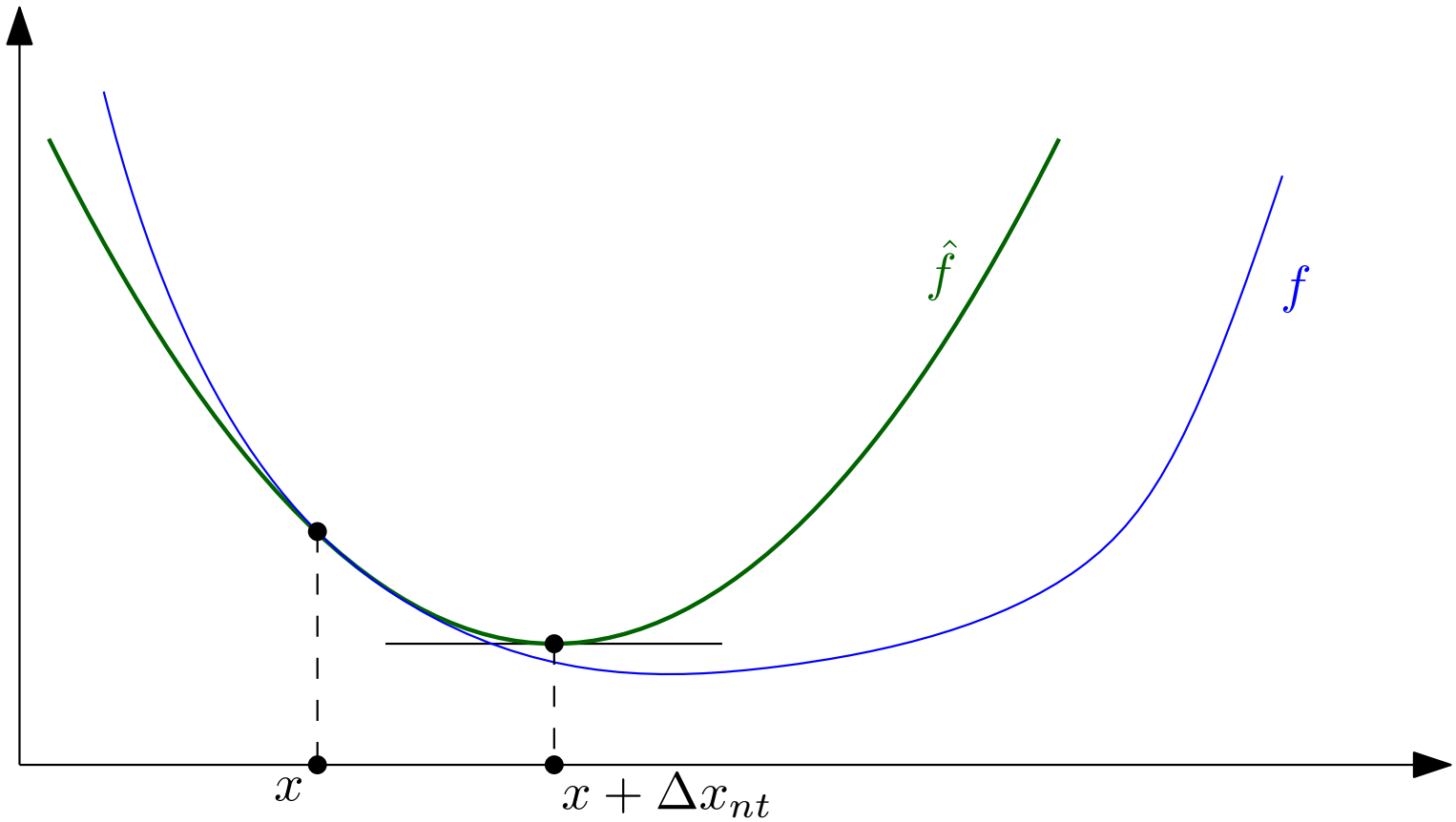
so $p^* = f(x^*)$.

Newton's Method: 1D Case



$$\hat{f}(x + \Delta x) = f(x) + f'(x)\Delta x + f''(x)\frac{\Delta x^2}{2}$$

Newton's Method: 1D Case



$$\hat{f}(x + \Delta x) = f(x) + f'(x)\Delta x + f''(x)\frac{\Delta x^2}{2}$$

Newton's Method: 1D Case

- $\hat{f}$ is the order-2 Taylor approximation of f at x.
- The *Newton step* is the value Δx_{nt} such that $\hat{f}(x + \Delta x_{nt})$ is minimum, so

$$\Delta x_{nt} = -\frac{f'(x)}{f''(x)}.$$

- The *Newton decrement* λ is given by

$$\lambda^2 = \frac{f'(x)^2}{f''(x)}.$$

  Then $\frac{1}{2}\lambda^2 = f(x) - \hat{f}(x + \Delta x_{nt})$ is the square of the difference between $f(x)$ and the minimum of $\hat{f}$.
- We regard $\frac{1}{2}\lambda^2$ as an approximation of the error made at the current step.

Newton's Method: 1D Case

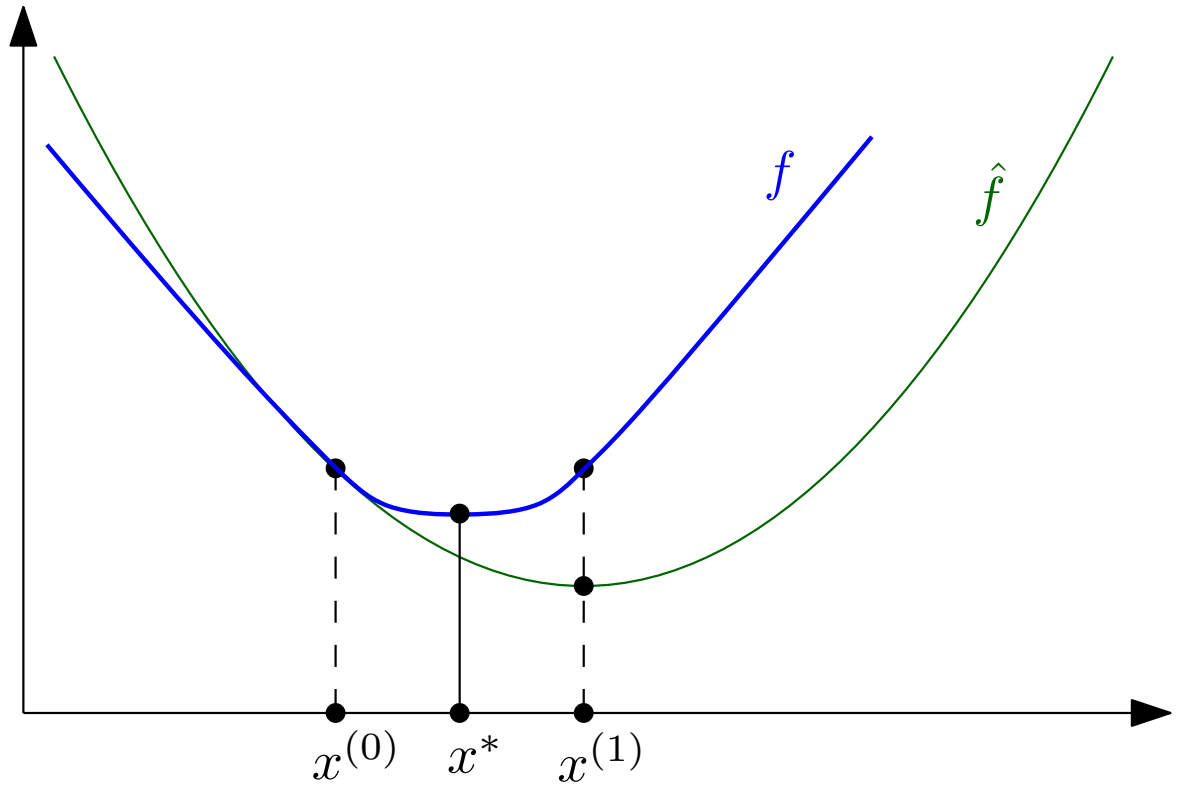
- We start from a point x in the domain of f, and we repeatedly add the Newton step to the current solution x, until the (approximate) error is smaller than ε.

Pseudocode

procedure PURE 1D NEWTON(f, x, ε)
  while true **do**
    $\Delta x_{nt} \leftarrow -f'(x)/f''(x)$ ▷ Newton step
    $\lambda^2 \leftarrow f'(x)^2/f''(x)$ ▷ Newton decrement
    if $\lambda^2/2 \leqslant \varepsilon$ **then** ▷ stopping criterion
      return x
    $x \leftarrow x + \Delta x_{nt}$ ▷ update

Newton's Method: 1D Case

- The pseudocode on previous slide gives the *pure* Newton's method.
- This method works well if we start close enough to the optimal.
- If not, it may diverge. For instance, it could traverse an infinite sequence $x^{(0)}$, $x^{(1)}$, $x^{(0)}$, $x^{(1)}$ ...

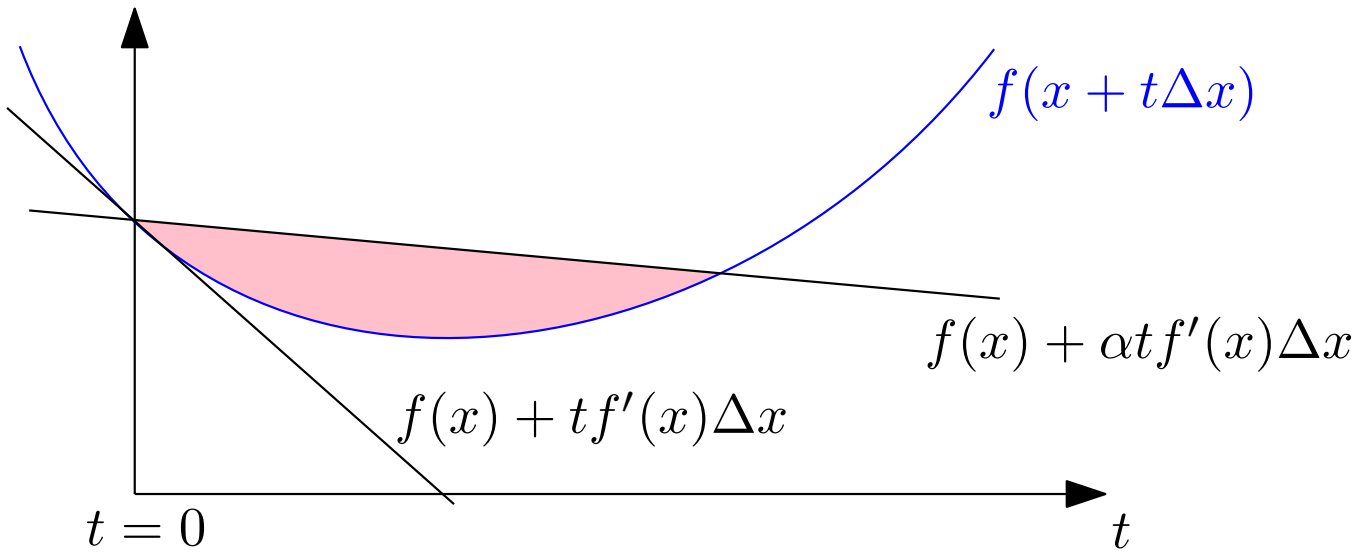


Newton's Method: 1D Case

- In order to ensure convergence, an initial stage using *backtracking line search* is needed. It uses two parameters $0 < \alpha < 0.5$ and $0 < \beta < 1$.

Pseudocode

1: **procedure** BACKTRACKING LINE SEARCH(f, x, Δx)
2: $t \leftarrow 1$
3: **while** $f(x + t\Delta x) > f(x) + \alpha t f'(x)\Delta x$ **do**
4: $t \leftarrow \beta t$
5: **return** t



Newton's Method: 1D Case

Pseudocode

1: **procedure** 1D NEWTON(f, x, ε)
2: **while** true **do**
3: $\Delta x_{nt} \leftarrow -f'(x)/f''(x)$ ▷ Newton step
4: $\lambda^2 \leftarrow f'(x)^2/f''(x)$ ▷ Newton decrement
5: **if** $\lambda^2/2 \leqslant \varepsilon$ **then** ▷ stopping criterion
6: **return** x
7: $t \leftarrow$ BACKTRACKING LINE SEARCH$(f, x, \Delta x_{nt})$
8: $x \leftarrow x + t\Delta x_{nt}$ ▷ update

Newton's Method: 1D Case

- Backtracking line search avoids overshooting.
- It can be proved that two stages occur:
- In the initial stage, backtracking yields $t < 1$.
- After this initial stage, it remains at $t = 1$ indefinitely.
- Newton's method converges very quickly: After the initial stage, the number of correct digits roughly *doubles* at each step.

Newton's Method: n-Dimensional Case

- The *gradient* ∇f and the *Hessian* $\nabla^2 f$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ are

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \qquad \nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

- The Hessian is often denoted H.
- The order-2 Taylor approximation of f at x is

$$\hat{f}(x+v) = f(x) + \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v \tag{1}$$

Newton's Method: n-Dimensional Case

Pseudocode

procedure NEWTON(f, x, ε)
  while true **do**
    $\Delta x_{nt} \leftarrow -\nabla^2 f(x)^{-1} \nabla f(x)$ ▷ Newton step
    $\lambda^2 \leftarrow \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$ ▷ Newton decrement
    if $\lambda^2/2 \leqslant \varepsilon$ **then** ▷ stopping criterion
      return x
    $t \leftarrow$ BACKTRACKING LINE SEARCH$(f, x, \Delta x_{nt})$
    $x \leftarrow x + t\Delta x_{nt}$ ▷ update

Pseudocode

procedure BACKTRACKING LINE SEARCH(f, x, Δx)
  $t \leftarrow 1$
  while $f(x + t\Delta x) > f(x) + \alpha t \nabla f(x)^T \Delta x$ **do**
    $t \leftarrow \beta t$
  return t

Newton's Method: n-Dimensional Case

- Same interpretation as in 1D: We move towards the minimum of the order-2 approximation.
- Backtracking line search is done along the direction of the Newton step.
- Analysis also shows very fast convergence in terms of number of steps.
- However each step can be expensive: Inversion of the Hessian.
  - In practice, the Newton step can be approximated faster.
- It cannot be used for linear programming, because LPs are *constrained* optimization problems.

Logarithmic Barrier Method

- We want to solve the following LP:

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \qquad i = 1, \ldots, m. \end{aligned}$$

- Using the matrix $A = (a_{ij})$ and the vectors $b = (b_i)$, $c = (c_j)$, and $x = (x_j)$, it can be written:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leqslant b \end{aligned}$$

Logarithmic Barrier Method

- We denote by a_i^T the ith row of A. Then the LP can also be written

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leqslant b_i, \qquad i = 1, \ldots, m. \end{array}$$

- The *indicator function* for the nonpositive reals $I_- : \mathbb{R} \to \bar{\mathbb{R}}$ is defined by:

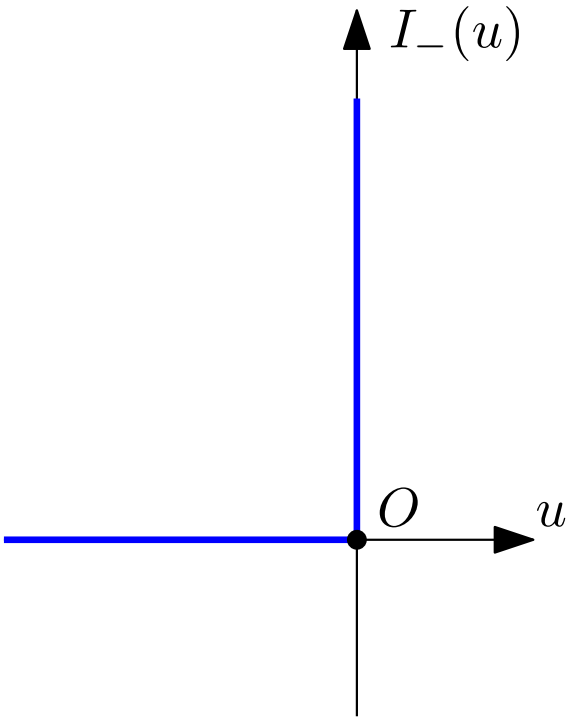
$$I_-(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ \infty & \text{if } x > 0 \end{cases}$$

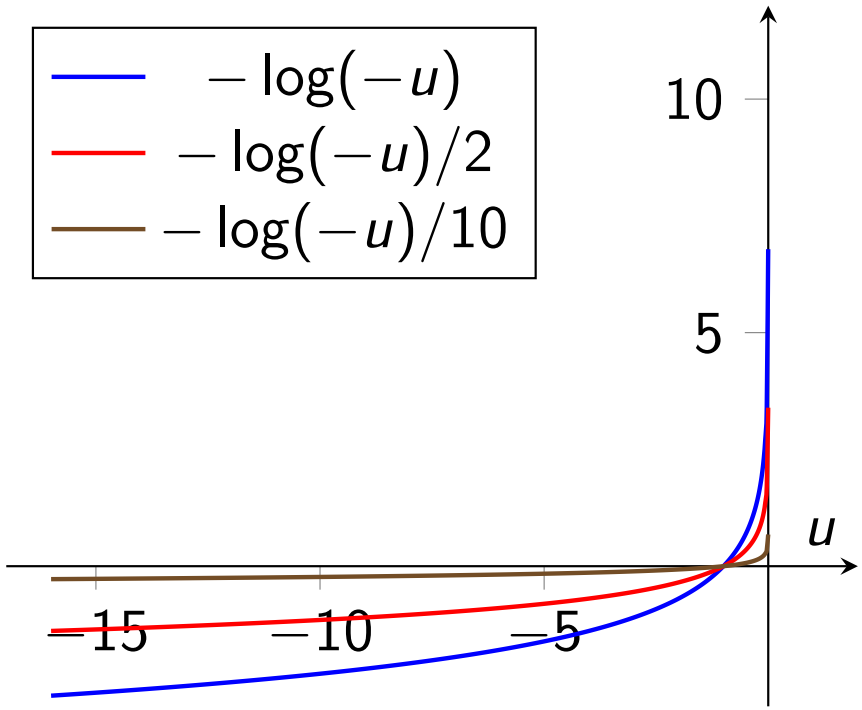
- Then the LP is equivalent to the *unconstrained* minimization problem

$$\text{minimize} \quad c^T x + \sum_{i=1}^{m} I_-(a_i^T x - b_i)$$

Logarithmic Barrier Method

- Problem: We cannot apply Newton's optimization method because I_- is not differentiable.
- So we approximate I_- by a function $\hat{I}_-(u) = -(1/t)\log(-u)$ where t is a parameter that sets the accuracy.





Logarithmic Barrier Method

- So the LP is approximated by the following optimization problem.

$$\text{minimize} \quad c^T x + \sum_{i=1}^{m} -\frac{1}{t} \log(b_i - a_i^T x)$$

  where the accuracy of the approximation improves when t increases.
- It is equivalent to

$$\text{minimize} \quad tc^T x - \sum_{i=1}^{m} \log(b_i - a_i^T x).$$

- It becomes an *unconstrained* minimization problem for a twice-differentiable convex function.
- So we can solve it with Newton's method.

Logarithmic Barrier Method

- We write $\Phi(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x)$, so the problem becomes

$$\text{minimize} \quad tc^T x + \Phi(x).$$

Let $x^*(t)$ be the optimal solution to this problem.

Interpretation

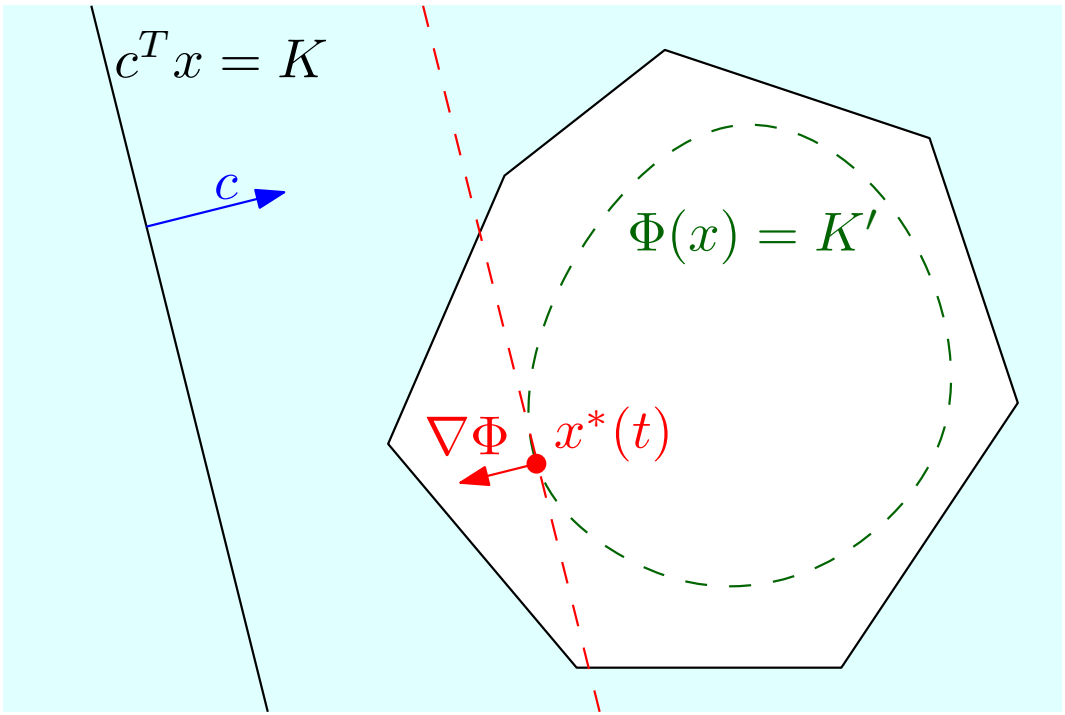
- The term $tc^T x$ pushes towards lower values of the original objective function (because we are minimizing).
- The term $\Phi(x)$ goes to ∞ when a constraint $a_i^T x \leqslant b_i$ becomes tight, which means that we approach the boundary of the feasible region.
- So the term $\Phi(x)$ pushes x inside the feasible region.
- When t increases, the term $tc^T x$ becomes more important, and $x^*(t)$ move towards an optimal solution of the original LP.

Logarithmic Barrier Method

- The gradient of the new objective function is

$$\nabla \left(tc^T x + \Phi(x) \right) = tc + \nabla \Phi(x)$$

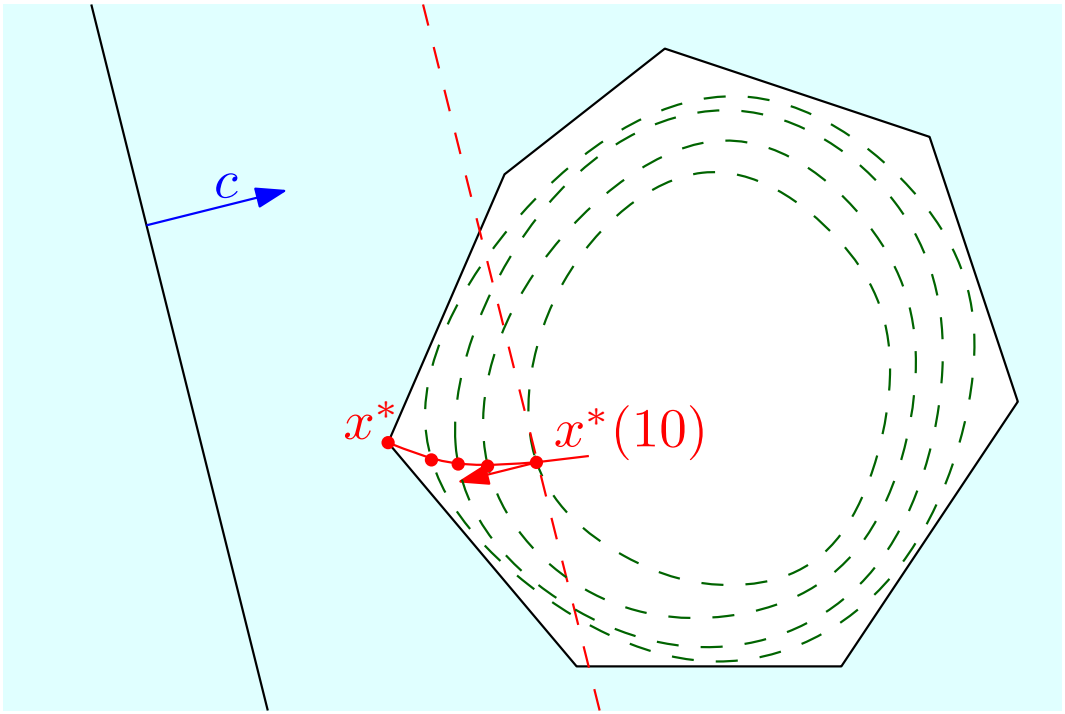
- So we have $tc + \nabla \Phi(x^*(t)) = 0$, and thus $\nabla \Phi$ points towards the opposite direction of c.



Logarithmic Barrier Method

- So the tangent at $x^*(t)$ to the level set containing $x^*(t)$ is parallel to any hyperplane $c^T x =$constant corresponding to the objective function of the LP.

Logarithmic Barrier Method



- When $t \to \infty$, the current solution $x^*(t)$ goes to the solution x^* of the original LP.
- The trajectory of $x^*(t)$ is called the *central path.*

Logarithmic Barrier Method

Pseudocode

1: **procedure** BARRIER METHOD(c, Φ, ε, x, t)
2: **while** true **do**
3: $x^*(t) \leftarrow$ NEWTON (f_t, x, ε) ▷ centering step
4: $x \leftarrow x^*(t)$
5: **if** $m/t < \varepsilon$ **then** ▷ stopping criterion
6: **return** x
7: $t \leftarrow \mu t$ ▷ increase t

- The input x should be feasible. Similar to the simplex method, we can find it with an auxiliary LP.
- f_t is the objective function for the current unconstrained minimization problem, so $f_t(u) = tc^T u + \Phi(u)$ for all u.
- $\mu > 1$ is a parameter. In practice $\mu = 10$ works well.

Logarithmic Barrier Method

- This algorithm only returns an *approximate solution*: The last point $\hat{x}$ visited by the barrier method satisfies $f(\hat{x}) - f(x^*) \leqslant \varepsilon$.
- So a final step called *termination* is needed, which finds x^* from $\hat{x}$.
- Idea:
  - Pick the n constraints that have the smallest slack.
  - x^* is at the intersection of their bounding hyperplanes.
  - It requires ε to be chosen small enough.
- Finally, we get the main result: (not proved in CSE515)

Theorem

The logarithm barrier method solves linear programming with m constraints in $O(m^3 L)$ time, where L is the total number of bits in the input.

Logarithmic Barrier Method

- The theorem above shows that there is a *weakly polynomial*-time algorithm for LP.
- It means polynomial in the input size, counting the number of bits of all input coefficients a_{ij}, b_i, c_j.

Open Problem

Does there exist a *strongly* polynomial algorithm for Linear Programming, that is, an algorithm whose running time is polynomial in n and m?

- This is one of the most important open problems in computer science.
- Possible approach: Find a pivoting rule for the simplex algorithm that makes a polynomial number of pivots.

Conclusion

- The logarithmic barrier method constructs a sequence of points in the *interior* of the feasible region, which converges to an optimal solution.
- So it is an *interior point* method.
- The simplex algorithm is very different: It walks along the *boundary* of the feasible region.
- Linear Programming can be solved (weakly) in polynomial time.
- So one way of finding a polynomial-time algorithm for a problem is to reduce it to LP. We saw several examples in the introductory lecture to LP.

Conclusion of this Lecture

- In practice, there are very efficient LP solvers, so it is a very useful tool.
- It can also be used to solve hard problems, by approximating the original problem using an LP. It sometimes provides a provable approximation, or it can be used as a heuristic. This is called *LP-Rounding*.
- Interior point methods can also be used for solving *convex optimization* problems: Minimizing a convex function over a convex set. (See reference book.)