

CSE515 Advanced Algorithms
Lecture 15
Introduction to Computational Complexity III

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Introduction

- Assignment 2 grades and solutions are posted.
- I will post assignment 3 on Friday. I will also post \LaTeX sources as some of you are typing solutions in \LaTeX .
- This is the last part of our 3-lectures introduction to computational complexity.
- I will present several **NP**-complete problems. For some of them, I will prove that they are **NP**-hard through a reduction.
- **Reference:** Chapter 34 of the textbook [Introduction to Algorithms](#) by Cormen, Leiserson, Rivest and Stein.

Clique

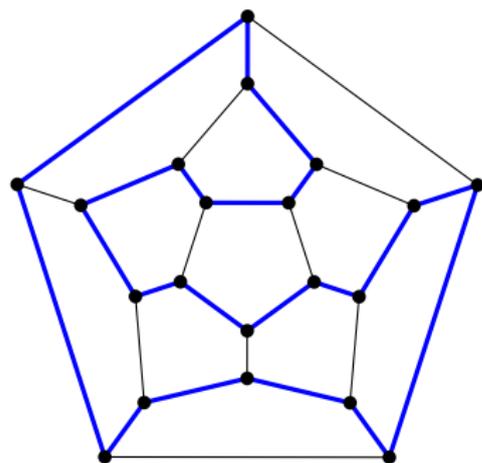
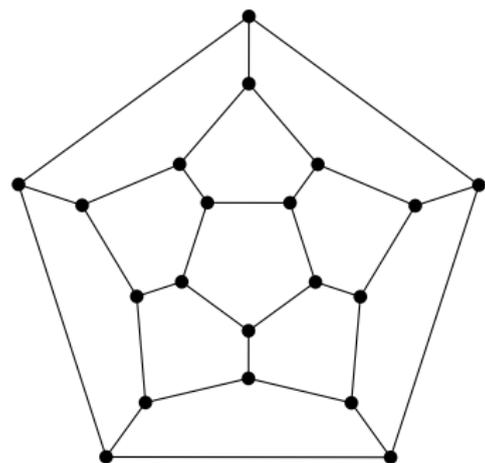
- In the previous two lectures, we saw that 3-SAT is **NP**-hard, $3\text{-SAT} \leq_p \text{VERTEX-COVER}$ and $\text{VERTEX-COVER} \leq_p \text{CLIQUE}$.
- It follows that **CLIQUE** is **NP**-hard.
- It is easy to see that **CLIQUE** is in **NP**: We can check in polynomial time whether a subset of vertices of the input graph forms a clique.
- Therefore:

Theorem

CLIQUE is **NP**-complete.

Hamiltonian Cycle

- Origin: A puzzle by Hamilton where the goal is to find a path that visits each city exactly once, and returns to the starting position, in the map below:



Hamiltonian Cycle

Definition

A *Hamiltonian cycle* of a graph is a cycle that visits each vertex exactly once.

Problem (HAM-CYCLE)

Given an input graph G , the *Hamiltonian cycle problem* is to decide whether G has a Hamiltonian cycle.

Theorem

The Hamiltonian cycle problem is **NP**-complete.

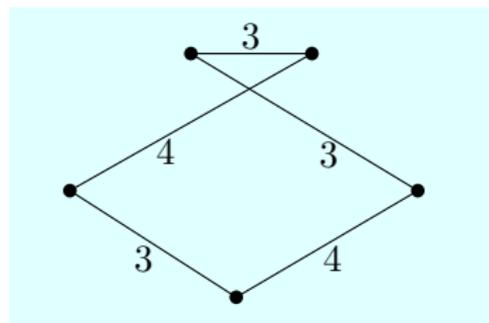
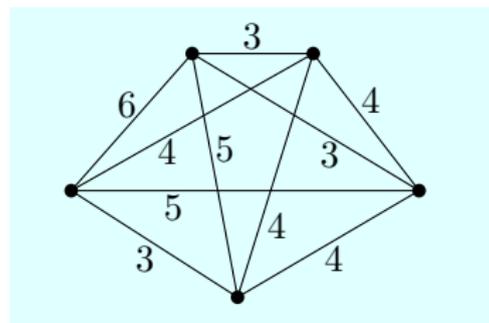
- The proof is not covered in this course. The textbook proves it by a reduction from vertex cover, i.e.

$$\text{VERTEX-COVER} \leq_p \text{HAM-CYCLE}.$$

The Traveling Salesman Problem (TSP)

Problem (TSP)

Given a set of cities, and the distance between each pair of cities, find the shortest tour.



- The input is given as an $n \times n$ matrix of distances $D = (d_{ij})$ where $d_{ij} = d_{ji} \geq 0$ and $d_{ii} = 0$ for all i, j .

The Traveling Salesman Problem (TSP)

- TSP is not in **NP**, because it is an optimization problem.
- Therefore we introduce:

Problem (DECIDE-TSP)

Given a TSP instance, and a number k , decide whether there is a tour of length at most k .

Proposition

DECIDE-TSP is **NP**-complete.

- Proof: see exercise set.

Corollary

TSP is **NP**-hard.

The Subset-Sum Problem

Problem (SUBSET-SUM)

Given a set S of integers and a target number t , decide whether there is a subset $S' \subseteq S$ whose elements sum to t , that is,

$$t = \sum_{x \in S'} x.$$

Example

Suppose that $S = \{1, 2, 7, 8, 16\}$.

- If $t = 11$ then it is a positive instance because $1 + 2 + 8 = 11$.
- If $t = 12$ then it is a negative instance.

The Subset-Sum Problem

Theorem

SUBSET-SUM is **NP**-complete.

Proof.

- SUBSET-SUM \in **NP** because we can verify in polynomial time whether a subset sums to t . (In fact it is trivially done in *linear* time.)
- SUBSET-SUM is **NP**-hard because $3\text{-SAT} \leq_p \text{SUBSET-SUM}$. The reduction is given in next slides: For any 3-SAT instance we construct in polynomial time an equivalent instance of SUBSET-SUM.

Detailed proofs are in the textbook.



The Subset-Sum Problem

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
ν_1	=	1	0	0	1	0	0	1
ν'_1	=	1	0	0	0	1	1	0
ν_2	=	0	1	0	0	0	0	1
ν'_2	=	0	1	0	1	1	1	0
ν_3	=	0	0	1	0	0	1	1
ν'_3	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s'_1	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s'_2	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s'_3	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s'_4	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

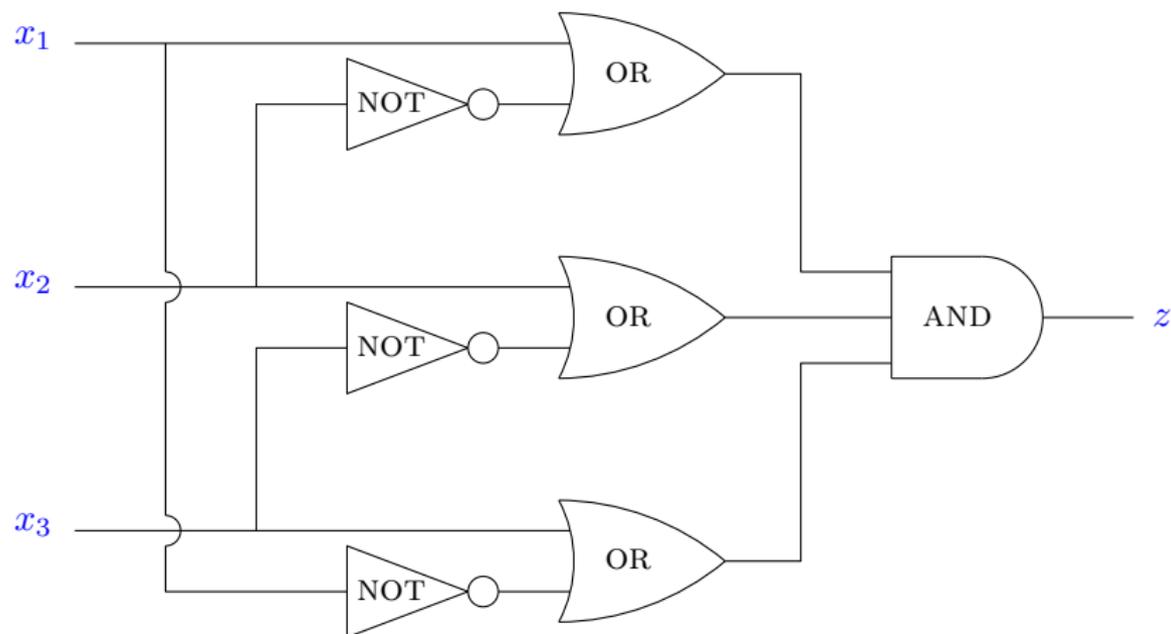
- $C_1 = x_1 \vee \neg x_2 \vee \neg x_3$
- $C_2 = \neg x_1 \vee \neg x_2 \vee \neg x_3$
- $C_3 = \neg x_1 \vee \neg x_2 \vee x_3$
- $C_4 = x_1 \vee x_2 \vee x_3$

The Subset-Sum Problem

	x_1	x_2	x_3	C_1	C_2	C_3	C_4
$\nu_1 =$	1	0	0	1	0	0	1
$\nu'_1 =$	1	0	0	0	1	1	0
$\nu_2 =$	0	1	0	0	0	0	1
$\nu'_2 =$	0	1	0	1	1	1	0
$\nu_3 =$	0	0	1	0	0	1	1
$\nu'_3 =$	0	0	1	1	1	0	0
$s_1 =$	0	0	0	1	0	0	0
$s'_1 =$	0	0	0	2	0	0	0
$s_2 =$	0	0	0	0	1	0	0
$s'_2 =$	0	0	0	0	2	0	0
$s_3 =$	0	0	0	0	0	1	0
$s'_3 =$	0	0	0	0	0	2	0
$s_4 =$	0	0	0	0	0	0	1
$s'_4 =$	0	0	0	0	0	0	2
$t =$	1	1	1	4	4	4	4

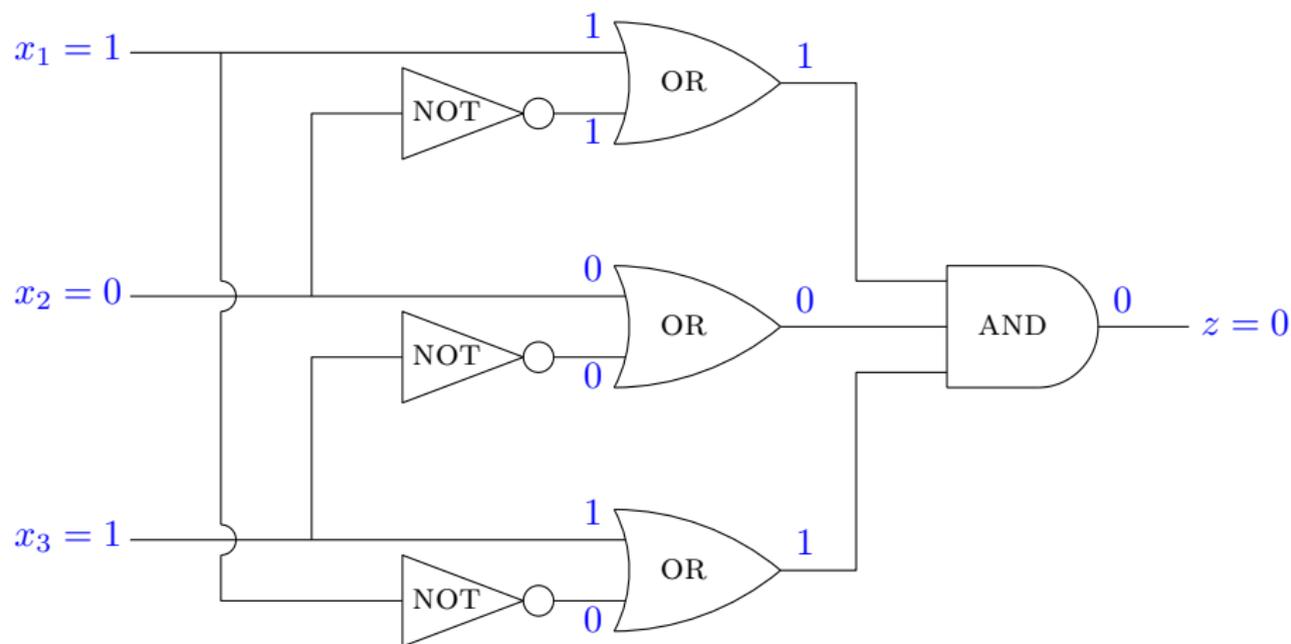
- $C_1 = x_1 \vee \neg x_2 \vee \neg x_3$
- $C_2 = \neg x_1 \vee \neg x_2 \vee \neg x_3$
- $C_3 = \neg x_1 \vee \neg x_2 \vee x_3$
- $C_4 = x_1 \vee x_2 \vee x_3$
- Satisfying assignment $x_1 = 0$, $x_2 = 0$, $x_3 = 1$
- The shaded rows, regarded as numbers in base 10, sum to t .

Circuit Satisfiability



- Circuit with input x_1 , x_2 , x_3 and output z .

Circuit Satisfiability



- On input $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, output is $z = 0$.

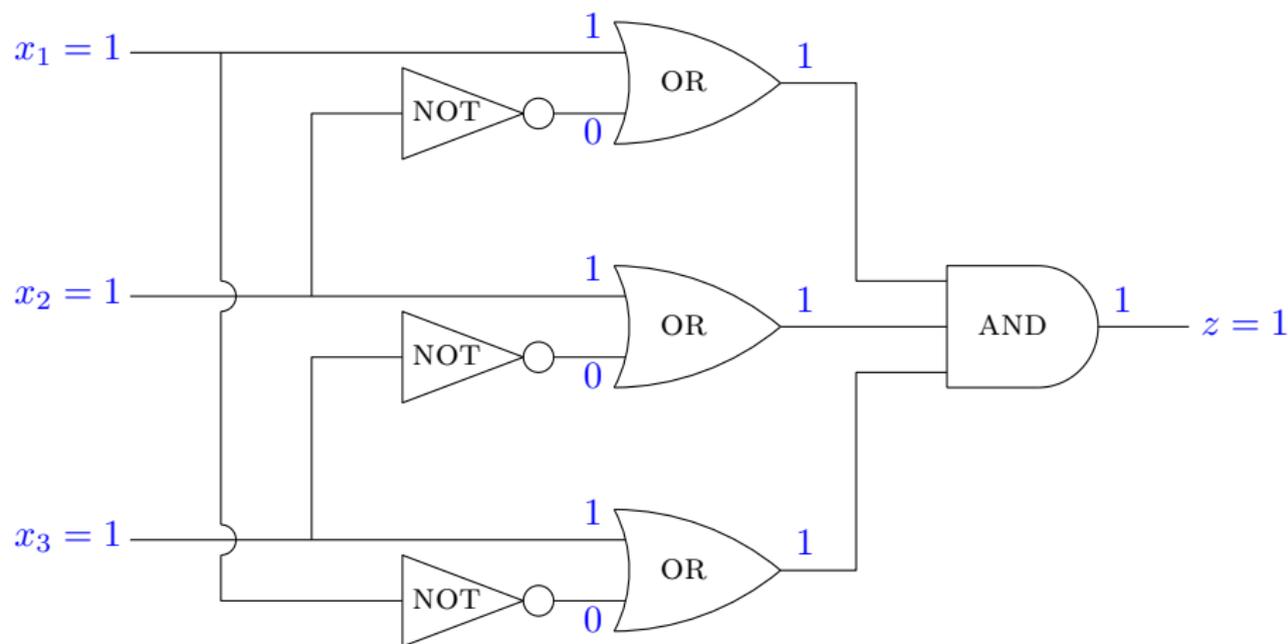
Circuit Satisfiability

Problem (CIRCUIT-SATISFIABILITY)

Given a Boolean circuit, decide whether there is an input such that the output is 1.

- Is the circuit above satisfiable?
- Yes: See next slide.

Circuit Satisfiability



- A satisfying assignment $x_1 = 1$, $x_2 = 1$, $x_3 = 1$

Circuit Satisfiability

Theorem

CIRCUIT-SATISFIABILITY is **NP**-complete.

Proof.

(Sketch) We use a reduction from 3-SAT. Any 3-SAT instance can be converted into an equivalent circuit in linear time. □

Example

The circuit above computes the (2-SAT) formula

$$(x_1 \vee \neg x_2) \wedge (x_2 \vee \neg x_3) \wedge (x_3 \vee \neg x_1).$$

Concluding Remarks on this Lecture

- A large variety of *combinatorial* problems are **NP**-complete. We saw graph algorithms problems, an arithmetic problem (SUBSET-SUM), a logic problem (3-SAT), a circuit problem . . .
- In order to prove that a problem is **NP**-hard, we usually prove a reduction from a known **NP**-hard problem.
- These reductions are often simple, but not always.
- First try to find a similar problem (i.e. HAM-CYCLE similar with TSP)
- Otherwise, 3-SAT is often used in these reductions.